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Three Point Spline Collocation Method for Solving General Linear and Nonlinear Eighth-Order Boundary-Value Problems

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☐ ABSTRACT ☐

In this paper, a spline collocation method is developed for finding numerical solutions of general linear eighth-order boundary-value problems (BVPs) and nonlinear eighth-order initial value problems (IVPs). The presented collocation method affords the spline solution by the polynomial of degree eleventh which satisfies the BVPs and IVPs at three collocation points. The study shows that the spline collocation method when is applied such this problems is existent and unique. Moreover, the purposed method if applied to these systems will be consistent and the global truncation error equal eleventh.

Numerical results are given for four examples to illustrate the implementation and efficiency of the method. Comparisons of the results obtained by the present method with results obtained by the other methods reveal that the present method is very effective and convenient.

Keywords: spline functions, collocation points, linear eighth-order BVPs, Rate of convergence, Error estimation.

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طريقة تجميع شرائحية بثلاث نقاط لحل مسائل القيم الحدية الخطية وغير الخطية المعممة من المرتبة الثامنة

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□ ملخّص □

يتم في هذا البحث تطوير طريقة شرائحية لإيجاد الحل العددي لمسائل القيم الحدية الخطية وغير الخطية في المعادلات التفاضلية المعممة من المرتبة الثامنة. الطريقة المقترحة تقدم الحل الشرائحي التقريبي باستخدام كثيرة حدود من الدرجة إحدى عشرة وتلك الحدودية تحقق المسائل الحدية والابتدائية المطروحة في ثلاث نقاط تجميع.

تبين الدراسة أن الطريقة المقترحة عندما تطبق لحل هذه المسائل تكون موجودة ومعرفة بشكل وحيد. كما تظهر الدراسة التحليلية أن الطريقة تكون متجانسة ومتقاربة وأن الخطأ المقتطع الشامل من الرتبة إحدى عشرة. تم اختبار الطريقة الشرائحية بحل أربع مسائل مختلفة، حيث تشير المقارنات لنتائج طريقتنا مع نتائج الطرائق الأخرى إلى أفضلية الطريقة المقترحة من حيث الدقة والفعالية.

الكلمات المفتاحية: دوال شرائحية، نقاط تجميع، مسائل القيمة الحدية الخطية من المرتبة الثامنة، معدل التقارب، تقدير الخطأ.

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Introduction:

Higher-order boundary value problems are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, engineering and applied physics. The several spline methods have been extensively applied in numerical ordinary differential equations due to its easy implementation and high-order accuracy. Recently, various powerful mathematical methods have been proposed to obtain numerical solutions for special linear eighth-order BVPs of the form:

$$y^{(8)}(x)+q(x)\ y(x)=g(x),\ \ x\in\ [a,b], \tag{1.1}$$

subject to the following two types of boundary conditions:

Type I: $$\begin{cases} y(a)=\alpha_0,\ y'(a)=\alpha_1, y''(a)=\alpha_2, y^{(3)}(a)=\alpha_3, \\ y(b)=\beta_0,\ y'(b)=\beta_1, y''(b)=\beta_2, y^{(3)}(b)=\beta_3 \end{cases}, \tag{1.1a}$$

Type II: $$\begin{cases} y(a)=\alpha_0,\ y''(a)=\alpha_2, y^{(4)}(a)=\alpha_4, y^{(6)}(a)=\alpha_6, \\ y(b)=\beta_0, y''(b)=\beta_2, y^{(4)}(b)=\beta_4, y^{(6)}(b)=\beta_6, \end{cases}, \tag{1.1b}$$

where α_i, β_i (i = 0, 1,3) and α_{2i}, β_{2i} (i = 2,3) are finite real constants and the functions $q(x)$ and $g(x)$ are continuous on $[a, b]$.

Several numerical methods including spline approximations and collocation [1-4], non-polynomial spline method by Jaleb and Farajeyan [1, 2008], similar spline technique by Rashidinia et al. in [2, 2009], spline collocation method by Lamnii et al. in [3, 2008] and quintic B-spline collocation method by Kasi et al. in [4, 2012] have been developed for solving the problem (1.1)-(1.1b). Noor and Mohyud-Din [5, 2007] have been implemented a analytical method is an combination of variational iteration method and decomposition method for solving (1.1) with conditions (1.1b). Variational iteration methods are been used by Siddiqi et al. [6, 2009] and Porshokouhi et al. [9, 2011] for solving (1.1) with conditions of the type (1.1a).

Moreover, optimal homotopy asymptotic and homotopy perturbation methods for the solution of higher-order boundary value problems are presented in [11-13].

In this paper, spline collocation techniques are presented of the numerical solutions for two types of problems. The first type, general linear eighth-order BVPs of the form:

$$y^{(8)}(x)+\sum_{i=0}^{7} q_i(x)\ y^{(i)}(x)=g(x),\ \ \ x\in\ \ [a,\ b], \tag{1.2}$$

subject to the following two cases of boundary conditions the type (1.1b) and (1.1a), where $q_i(x)$ $(i=0,...,7)$ are all continuous functions on $[a, b]$.

The second type, general nonlinear eighth-order initial value problems of the form:

$$y^{(8)}(x)=f(x,y,y',y''...,y^{(7)})\ ,\ x\in\ [a,b], \tag{1.3}$$

with the following initial conditions:

$$y^{(i)}(a)=\alpha_i,\ i=0,1,...,7\,. \tag{1.3a}$$

Hesaaraki and Jalilian [7, 2008] and Torvattanabun and Koonprasert [8, 2010] have been applied variational iteration methods of numerical solution for the problem (1.3) with boundary conditions (1.1a). Fazal-i-Haq Fazal-i-Haq et al. [10, 2010] have been presented a collocation method based on Haar wavelets of the numerical solution for the problems from the form (1.3)-(1.3a).

Importance of Research and its Aim:

It is well known that the analysis solutions of those higher-order BVPs are either very difficult or not existent. For these causes, the numerical solutions of proposed problems are very important.

This paper aims to develop spline polynomial method with three collocation points for finding the spline solutions for general linear eighth-order BP (1.2), and general nonlinear eighth-order IVPs (1.3).

Methodology:

The paper is organized as follows. In **section** 2, the eighth-order BVP (1.2) with two types of boundary conditions (1.1a)-(1.1b) is transformed into five initial value problems (IVPs). Spline functions with three collocation points are directly applied into eighth-order IVPs and then finding the numerical spline solution and its derivatives up to eighth-order of the problem (1.2) with the two cases of boundary conditions (1.1a)-(1.1b). Moreover, spline functions are directly applied into nonlinear eighth-order IVPs for finding its spline numerical solution. **Section** 3, the existence and uniqueness of spline solution of the eighth-order BVP are proved. The error estimation and order of convergence of the spline method are discussed in **section 4**. **Section 5**, examples and comparisons are made to confirm the efficiency and implementation of the proposed method.

2- Spline Collocation Method

In this section, the eighth-order BVP is transformed into five IVPs. After that, spline functions are formulated to be applied directly into the five IVPs for finding the spline solution and its derivatives up to eighth-order of the problem proposed.

2.1 Solution Scheme of eighth-order BVP

Consider the eighth-order BVP (1.2):

$$y^{(8)}(x) = -\sum_{i=0}^{7} q_i(x)\, y^{(i)}(x) + g(x), \quad x \in [a, b], \tag{2.1}$$

subject to the following two cases of boundary conditions:

Case I: $$\begin{cases} y(a)=\alpha_0,\ y'(a)=\alpha_1, y''(a)=\alpha_2, y^{(3)}(a)=\alpha_3, \\ y(b)=\beta_0,\ y'(b)=\beta_1, y''(b)=\beta_2, y^{(3)}(b)=\beta_3 \end{cases}, \tag{2.1a}$$

Case II: $$\begin{cases} y(a)=\alpha_0,\ y''(a)=\alpha_2, y^{(4)}(a)=\alpha_4, y^{(6)}(a)=\alpha_6, \\ y(b)=\beta_0, y''(b)=\beta_2, y^{(4)}(b)=\beta_4, y^{(6)}(b)=\beta_6, \end{cases}, \tag{2.1b}$$

Let $y(x)$ be the unique solution to the BVP (2.1) with either conditions case (2.1a) or (2.1b), then this solution is associated by a linear combination consist of five special IVPs. To find it, we assume that $U(x)$ is the unique solution to the following eighth-order IVP:

$$U^{(8)}(x) = -\sum_{i=0}^{7} q_i(x) U^{(i)}(x) + g(x), \quad a \le x \le b\ , \tag{2.2}$$

with the following initial conditions:

Case I: $$U^{(k)}(a)=\alpha_k,\ (k=0,1,2,3),\ U^{(k)}(a)=0,\ (k=4,...,7), \tag{2.2a}$$

Case II: $$U^{(2k)}(a)=\alpha_{2k},\ U^{(2k+1)}(a)=0,\ (k=0,1,2,3) \tag{2.2b}$$

In addition, suppose that $U_1(x)$, $U_2(x)$ $U_3(x)$ and $U_4(x)$ are the unique solutions to the following four homogeneous eighth-order IVPs, respectively, the first equation:

$$U_1^{(8)} = -\sum_{i=0}^{7} q_i(x) U_1^{(i)}(x), \quad a \le x \le b, \tag{2.3}$$

with the following initial conditions:

Case I: $U_1^{(k)}(a) = 0, \; (k = 0,1,2,3), \; U_1^{(4)}(a) = 1, \; U_1^{(k)}(a) = 0 \; (k = 5,6,7),$ (2.3a)

Case II: $U_1^{(2k)}(a) = 0 \, (k = 0,1,2,3), \; U_1'(a) = 1, U_1^{(2k+1)}(a) = 0, \; (k = 1,2,3)$. (2.3b)

The second equation

$$U_2^{(8)} = -\sum_{i=0}^{7} q_i(x) U_2^{(i)}(x), \quad a \le x \le b \tag{2.4}$$

with the following initial conditions:

Case I: $U_2^{(k)}(a) = 0, \; (k = 0,1,\dots,4), \; U_2^{(5)}(a) = 1, \; U_2^{(k)}(a) = 0 \; (k = 6,7),$ (2.4a)

Case II: $U_2^{(2k)}(a) = 0 \, (k = 0,1,2,3), \; U_2'(a) = 0, U_2^{(3)}(a) = 1, U_2^{(2k+1)}(a) = 0, \; (k = 2,3)$ (2.4b)

The third equation

$$U_3^{(8)} = -\sum_{i=0}^{7} q_i(x) U_3^{(i)}(x), \quad a \le x \le b, \tag{2.5}$$

with the following initial conditions:

Case I: $U_3^{(k)}(a) = 0, \; (k = 0,1,\dots,5), \; U_3^{(6)}(a) = 1, \; U_3^{(7)}(a) = 0,$ (2.5a)

Case II: $U_3^{(2k)}(a) = 0 \, (k = 0,1,2,3), \; U_3^{(5)}(a) = 1, \; U_3^{(7)}(a) = U_3'(a) = U_3^{(3)}(a) = 0$. (2.5b)

The final fourth equation:

$$U_4^{(8)} = -\sum_{i=0}^{7} q_i(x) U_4^{(i)}(x), \quad a \le x \le b \tag{2.6}$$

with only the following initial conditions:

$U_4^{(k)}(a) = 0, \; (k = 0,1,\dots,6), \; U_4^{(7)}(a) = 1,$ (2.6a)

Then, for four real constants c_1, c_2, c_3, and c_4 there exist the linear combinations, such that:

$$y(x) = U(x) + \sum_{k=1}^{4} c_k U_k \tag{2.7}$$

is a solution to the eighth-order BVP (2.1) with either conditions case (2.1a) or case (2.1b), as seen by the following computations:

$$y^{(8)}(x) = U^{(8)}(x) + \sum_{k=1}^{4} c_k U_k^{(8)} =$$

$$= -\sum_{i=0}^{7} q_i(x) U^{(i)}(x) + g(x) + \sum_{k=1}^{4} c_k \left[-\sum_{i=0}^{7} q_i(x) U_k^{(i)}(x) \right]$$

$$= -\sum_{i=0}^{7} q_i(x) \left[U^{(i)}(x) + \sum_{k=1}^{4} c_k U_k^{(i)}(x) \right] + g(x) = -\sum_{i=0}^{7} q_i(x) \; y^{(i)}(x) + g(x),$$

where $y^{(i)}(x) = U^{(i)}(x) + \sum_{k=1}^{4} c_k U_k^{(i)}(x), i = 0,1,\dots,7$.

Now, it will be illustrated that the solution $y(x)$ the formulated by equation (2.7) holds on the two cases of boundary values (2.1a)-(2.1b), thus from conditions the case (2.1a) it yields out:

$$y^{(i)}(a) = U^{(i)}(a) + \sum_{k=1}^{4} c_k \, U_k^{(i)}(a) = \alpha_i + \sum_{k=1}^{4} c_k (0) = \alpha_i, \quad (i = 0,1,2,3)$$

The unknown constants c_1, c_2, c_3, and c_4 will be determined from the remainder of the end conditions by solving the system of equations:

$$y^{(i)}(b) = U^{(i)}(b) + \sum_{k=1}^{4} c_k U_k^{(i)}(b) \equiv \beta_i, \quad (i = 0,1,2,3), \tag{2.8}$$

Moreover, from the second case conditions (2.1b), it also produces:

$$y^{(2i)}(a) = U^{(2i)}(a) + \sum_{k=1}^{4} c_k U_k^{(2i)}(a) = \alpha_{2i} + \sum_{k=1}^{4} c_k(0) = \alpha_{2i}, \quad (i = 0,1,2,3) ,$$

The unknown constants c_1, c_2, c_3, and c_4 will are determined from the remainder end conditions of the second case (2.1b), namely, the system of equations:

$$y^{(2i)}(b) = U^{(2i)}(b) + \sum_{k=1}^{4} c_k U_k^{(2i)}(b) \equiv \beta_{2i}, \quad (i = 0,1,2,3), \tag{2.9}$$

Now, since the proposed BVP (2.1)-(2.1b) has reduced into five IVPs (2.2)-(2.6), spline solutions with three collocation points are applied for solving of eighth-order IVPs.

2.2 Formulation of the Spline Approximations.

Denote by $x_i = a + i\,h,\ i = 0(1)N$, the grid points of the uniform partition of $[a,b]$ into subintervals $I_k = [x_k, x_{k+1}],\ k = 0,1,\ldots,N-1$, and $h = (b-a)/N$ is the constant stepsize. Let $S(x)$ be the spline approximation to the function $y(x)$ can be represented on each I_k by:

$$S(x) = \sum_{i=0}^{8} \frac{(x-x_k)^i}{i!} S_k^{(i)} + \sum_{i=9}^{11} \frac{(x-x_k)^i}{i!} C_{k,i-8}\ , x \in [x_k, x_{k+1}],\ k = 0,\ldots,N-1 \ , \tag{2.10}$$

where $\quad S^{(i)}(a) = S_0^{(i)}\ \ (i = 0,\ldots,8)$.

The presented method uses three collocation points:

$$x_{k+z_j} = x_k + h\, z_j\,,\ (j=1,2,3), \tag{2.10a}$$

such that

$$0 < z_1 < z_2 < z_3 = 1$$

By applying the spline approximation (2.10) and its derivatives up to eighth-order with respect to x, into eighth-order IVPs (2.2)-(2.6), to be satisfied with three collocation points (2.10a), in each subinterval $I_k = [x_k,\ x_{k+1}]$, k=0(1)N-1, then we have, respectively:

$$S_U^{(8)}(x_{k+z_j}) = -\sum_{i=0}^{7} g_i(x_{k+z_j}) S_U^{(i)}(x_{k+z_j}) + g(x_{k+z_j}) \ , \quad j = 1,2,3, \quad k = 0(1)N-1, \tag{2.11}$$

with the following initial conditions:

Case I: $\ S_U^{(i)}(a) = \alpha_i\ \ (i = 0,1,2,3), \quad S_U^{(i)}(a) = 0\ \ (i = 4,5,6,7)$, (2.11a)

Case II: $\ S_U^{(2i)}(a) = \alpha_{2i} \quad S_U^{(2i+1)}(a) = 0\ \ (i = 0,1,2,3)$. (2.11b)

$$S_{U_1}^{(8)}(x_{k+z_j}) = -\sum_{i=0}^{7} q_i(x_{k+z_j}) S_{U_1}^{(i)}(x_{k+z_j}) \ , \quad j = 1(1)3, \quad k = 0(1)N-1, \tag{2.12}$$

with the following initial conditions:

Case I: $\ S_{U_1}^{(i)}(a) = 0\ \ (i = 0,1,2,3),\ \ S_{U_1}^{(4)}(a) = 1,\ \ S_{U_1}^{(i)}(a) = 0\ \ (i = 5,6,7)$, (2.12a)

Case II: $\ S_{U_1}^{(2i)}(a) = 0\ \ (i = 0,1,2,3),\ \ S_{U_1}^{(1)}(a) = 1,\ \ S_{U_1}^{(2i+1)}(a) = 0\ \ (i = 1,2,3)$. (2. 12b)

$$S_{U_2}^{(8)}(x_{k+z_j}) = -\sum_{i=0}^{7} q_i(x_{k+z_j}) S_{U_2}^{(i)}(x_{k+z_j}) \ , \quad j = 1(1)3, \quad k = 0(1)N-1, \tag{2.13}$$

with the following initial conditions:

Case I: $S_{U_2}^{(i)}(a)=0\ (i=0,1,...,4),\ S_{U_2}^{(5)}(a)=1,\ S_{U_2}^{(i)}(a)=0\ \ (i=6,7)$, (2.13a)

Case II: $S_{U_2}^{(2i)}(a)=0\ (i=0,1,2,3),\ S_{U_2}^{(1)}(a)=0,\ S_{U_2}^{(3)}(a)=1,\ S_{U_2}^{(2i+1)}(a)=0\ (i=2,3)$. (2. 13b)

$$S_{U_3}^{(8)}(x_{k+z_j})=-\sum_{i=0}^{7}q_i(x_{k+z_j})S_{U_3}^{(i)}(x_{k+z_j})\ ,\quad j=1(1)3,\quad k=0(1)N-1, \qquad (2.14)$$

with the following initial conditions:

Case I: $S_{U_3}^{(i)}(a)=0\ (i=0,1,...,5),\ S_{U_3}^{(6)}(a)=1,\ S_{U_3}^{(7)}(a)=0$, (2.14a)

Case II: $S_{U_3}^{(i)}(a)=0\ (i=0,1,...,3),\ S_{U_3}^{(5)}(a)=1,\ S_{U_3}^{(i)}(a)=0\ \ (i=6,3)$. (2.14b)

Finally, applying to five IVP:

$$S_{U_4}^{(8)}(x_{k+z_j})=-\sum_{i=0}^{7}q_i(x_{k+z_j})S_{U_4}^{(i)}(x_{k+z_j})\ ,\quad j=1(1)3,\quad k=0(1)N-1, \qquad (2.15)$$

with only the following initial conditions:

$S_{U_4}^{(i)}(a)=0\ (i=0,1,...,6),\ S_{U_4}^{(7)}(a)=1$. (2.15a)

Now, by substituting spline solutions to the system of linear equations (2.8), the coefficients c_1,c_2,c_3, and c_4 the associated by boundary conditions of case I , will be known as follow:

$$\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}S_{U_1}(b) & S_{U_2}(b) & S_{U_3}(b) & S_{U_4}(b)\\ S'_{U_1}(b) & S'_{U_2}(b) & S'_{U_3}(b) & S'_{U_4}(b)\\ S''_{U_1}(b) & S''_{U_2}(b) & S''_{U_3}(b) & S''_{U_4}(b)\\ S_{U_1}^{(3)}(b) & S_{U_2}^{(3)}(b) & S_{U_3}^{(3)}(b) & S_{U_4}^{(3)}(b)\end{bmatrix}^{-1}\begin{bmatrix}\beta_0-S_U(b)\\ \beta_1-S'_U(b)\\ \beta_2-S''_U(b)\\ \beta_3-S_U^{(3)}(b)\end{bmatrix}$$

Also, by substituting spline solutions to the system of linear equations (2.9), the coefficients c_1,c_2,c_3, and c_4 for boundary conditions of case II , yield out:

$$\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}S_{U_1}(b) & S_{U_2}(b) & S_{U_3}(b) & S_{U_4}(b)\\ S''_{U_1}(b) & S''_{U_2}(b) & S''_{U_3}(b) & S''_{U_4}(b)\\ S_{U_1}^{(4)}(b) & S_{U_2}^{(4)}(b) & S_{U_3}^{(4)}(b) & S_{U_4}^{(4)}(b)\\ S_{U_1}^{(6)}(b) & S_{U_2}^{(6)}(b) & S_{U_3}^{(6)}(b) & S_{U_4}^{(6)}(b)\end{bmatrix}^{-1}\begin{bmatrix}\beta_0-S_U(b)\\ \beta_2-S''_U(b)\\ \beta_4-S_U^{(4)}(b)\\ \beta_6-S_U^{(6)}(b)\end{bmatrix}$$

Thus, the spline solutions $S^{(i)}(x), i=0,...,8$ of the BVP (2.1) with two cases of boundary conditions (2.1a)-(2.1b), will be known by:

$$S^{(i)}(x_k)=S_U^{(i)}(x_k)+\sum_{j=1}^{4}c_jS_{U_j}^{(i)}(x_k),\quad i=0,1,...,8. \qquad (2.16)$$

Moreover, applying collocation points $x_{k+z_j}=x_k+h\,z_j$, (j=1,2,3) to (2.10), we obtain

$$S(x_{k+z_j})=\sum_{i=0}^{8}\frac{(h\,z_j)^i}{i!}S_k^{(i)}+\sum_{i=9}^{11}\frac{(h\,z_j)^i}{i!}C_{k,i-8},\quad j=1,..,3,\quad k=0,...,N-1. \qquad (2.17)$$

where $z_j=j/3$, $x_{k+z_j}\in]x_k,x_{k+1}]$, ($j$=1,2,3).

The first three coefficients $C_{k,1},C_{k,2},C_{k,3}$ are computed from the linear systems (2.11)-(2.15) by using the initial value conditions if k=0, or from the previous steps if k>1.

2.3 Spline Solution of nonlinear eighth-order IVP

The numerical solutions of nonlinear eighth-order IVPs by presented spline method are more easy than BVPs, because the spline approximation (2.10) and its derivatives $S^{(i)}(x), i=0,...,8$, will be applied directly without reducing the problem to system of first-order differential equations. Now, spline collocation method is applied into (1.3)-(1.3a), to be satisfied with collocation points (2.10a), in each subinterval $\mathrm{I}_k=[x_k, x_{k+1}]$, as follow:

$$S^{(8)}(x_{k+z_j})=f[x_{k+z_j}, S(x_{k+z_j}), S'(x_{k+z_j}),..., S^{(7)}(x_{k+z_j})], \quad x_{k+z_j}\in I_k, \qquad k=0(1)N\text{-}1$$

with the following initial conditions:

$S_0^{(i)}(a)=\alpha_i,\ i=0,1,...,7$.

2.4 A unique Spline Collocation Solution

Consider the following linear eighth-order IVP:

$$\begin{cases} y^{(8)}(x)=F[x, y(x),..., y^{(7)}(x)], \ \ x\in[a,b] \\ y^{(d)}(a)=S_0^{(d)},\ \ d=0(1)7. \end{cases} \tag{2.18}$$

Suppose that $F:[a, b]\times C[a, b]\times\cdots\times C^7[a, b]\to R$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last argument:

$$|F(x, y_0,..., y_7)-F(x, \ddot{y}_0,..., \ddot{y}_7)|\le L\sum_{i=0}^{7}|y_i-\ddot{y}_i|, \forall\ (x, y_0,..., y_7), (x, \ddot{y}_0,..., \ddot{y}_7)\in[a, b]\times R^8$$

where the constant L is called a Lipschitz constant for F.

These conditions assure the existence of a unique solution $y(x)$ of problem (2.18).

By applying the Spline approximations (2.10) and its derivatives into the problem (2.18), to be satisfied with three collocation points (2.10a), we obtain the linear system:

$$S_k^{(8)}+(h z_j)C_{k,1}+\frac{(h z_j)^2}{2}C_{k,2}+\frac{(h z_j)^3}{3!}C_{k,3}=F(x_{k+z_j}, S(x_{k+z_j}),..., S^{(7)}(x_{k+z_j})), \tag{2.19}$$

$$j=1,..,3, \quad k=0,..., N-1,$$

$$S^{(d)}(a)=S_0^{(d)},\ \ d=0(1)7. \tag{2.20}$$

We rewrite (2.19) in the matrices formula:

$$A\overline{C}_k=\hat{F}_k-\hat{S}_k \tag{2.21}$$

where

$$A=\begin{bmatrix} h z_1 & \frac{h^2 z_1^2}{2!} & \frac{h^3 z_1^3}{3!} \\ h z_2 & \frac{h^2 z_2^2}{2!} & \frac{h^3 z_2^3}{3!} \\ h & \frac{h^2}{2!} & \frac{h^3}{3!} \end{bmatrix}, \ \overline{C}_k=\begin{bmatrix} C_{k,1} \\ C_{k,2} \\ C_{k,3} \end{bmatrix}, \ \hat{F}_k=\begin{bmatrix} F_{k+z_1} \\ F_{k+z_2} \\ F_{k+1} \end{bmatrix}, \hat{S}_k=\begin{bmatrix} S_k^{(8)} \\ S_k^{(8)} \\ S_k^{(8)} \end{bmatrix},$$

$$F_{k+Z_j}=F[x_{k+Z_j}, S(x_{k+Z_j}),\ldots, S^{(7)}(x_{k+Z_j})], \quad j=1,2,3.$$

Theorem : Suppose that $F\in C([\mathrm{a},\mathrm{b}]\times\Re)$ satisfies Lipschitz condition, and if

$$h^6<\frac{26244*7!}{6457\ L} \tag{2.22}$$

then the spline approximation solution S(x) exists and is uniquely defined for $z_j = j/3$ (j=1,2,3), where L is a Lipschitz constant for F.

Proof. It is sufficient to prove that the vector $\overline{C}_k$ can be uniquely determined for arbitrary given $\overline{S}_k$. We note that if $z_j = j/3$ (j=1,2,3), then the matrix A^{-1} exists and is nonsingular because $Det(A) = h^6/729$.

Let $\overline{C}_k^1, \overline{C}_k^2 \in R^3$, then using $\|.\|_1$ from (2.21), we have

$\overline{C}_k^1 = A^{-1}\hat{F}_k^1 - A^{-1}\hat{S}_k$ and $\overline{C}_k^2 = A^{-1}\hat{F}_k^2 - A^{-1}\hat{S}_k$

Thus $\overline{C}_k^1$ and $\overline{C}_k^2$ can be written in the form

$\overline{C}_k^1 = \overline{Q}_k(C_{k,1}^1, C_{k,2}^1, C_{k,3}^1, h)$ and $\overline{C}_k^2 = \overline{Q}_k(C_{k,1}^2, C_{k,2}^2, C_{k,3}^2, h)$

Applying $\|.\|_1$, Lipschitz condition and using *Mathematica*, we get

$$\|\overline{Q}_k(\overline{C}_k^1) - \overline{Q}_k(\overline{C}_k^2)\| = \|(A^{-1}\hat{F}_k^1) - (A^{-1}\hat{F}_k^2)\| = \|A^{-1}(\hat{F}_k^1 - \hat{F}_k^2)\| \le$$

$$\{L_1 H_1 |C_{k,1}^1 - C_{k,1}^2| + L_2 H_2 |C_{k,2}^1 - C_{k,2}^2| + L_3 H_3 |C_{k,3}^1 - C_{k,3}^2|\} \le$$

$$\{L_1(\frac{6457}{26244*7!}h^6)|C_{k,1}^1 - C_{k,1}^2| + L_2(\frac{14879}{590490*7!}h^7)|C_{k,2}^1 - C_{k,2}^2| +$$

$$L_3(\frac{2239}{13365*9!}h^8)|C_{k,3}^1 - C_{k,3}^2|\}$$

$$\le L(\frac{6457}{26244*7!}h^6)\{|C_{k,1}^1 - C_{k,1}^2| + |C_{k,2}^1 - C_{k,2}^2| + |C_{k,3}^1 - C_{k,3}^2|\},$$

where

$$A^{-1} = \begin{bmatrix} \frac{9}{h} & -\frac{9}{2h} & \frac{1}{h} \\ -\frac{45}{h^2} & \frac{45}{h^2} & -\frac{9}{h^2} \\ \frac{81}{h^3} & -\frac{81}{h^3} & \frac{27}{h^3} \end{bmatrix}, \qquad (2.23)$$

$$H_1 = \frac{605}{8817984}h^6 - \frac{7}{314928}h^7 + \frac{23}{9447840}h^8 \le \frac{6457}{26244*7!}h^6, \quad \forall\, h \in]0,1[,$$

$$H_2 = \frac{311}{44089920}h^7 - \frac{27479}{11904278400}h^8 + \frac{121}{476171136}h^9 \le \frac{14879}{590490*7!}h^7, \quad \forall\, h \in]0,1[,$$

$$H_3 = \frac{2591}{10935*9!}h^8 - \frac{4}{18600435}h^9 + \frac{311}{2598156*7!}h^{10} \le \frac{2239}{13365*9!}h^8, \quad \forall\, h \in]0,1[,$$

$$L = \max(L_1, L_2, L_3), \quad H_1 = \max(H_1, H_2, H_3) \le \frac{6457}{26244*7!}h^6, \ \forall\, h \in]0,1[.$$

Thus, the function $\overline{Q}_k$ defines a contraction mapping if $h^6 L \frac{6457}{26244*7!} < 1$ which satisfies (2.22). Hence there exists a unique $\overline{C}_k$ that satisfies $\overline{C}_k = \overline{Q}_k(C_{k,1}, C_{k,2}, C_{k,3}, C_{k,4}, h)$ which may be found by iterations $\overline{C}_k^{p+1} = \overline{Q}_k(\overline{C}_k^p,, h)$, p=0,1,2,… and this completes the proof.

3 Error estimation and convergence

We assume that $y(x)\in C^{12}[a,b]$, the unique solution of the linear eighth-order BVP and S(x) be a spline approximation solution to y(x), also $T=(\bar{\tau}_k)$ is a 3-dimensional column vector. Here, the vector $\bar{\tau}_k$ is the local truncation error. Applying the Spline solution $S(x)$ on three collocation points $x_{k+z_j}=x_k+z_j\,h$, (j=1,2,3), putting $y(x_{k+z_j})=y(x_k+hz_j)$, $S_k^{(m)}=S^{(m)}(x_k)$ and $y_k^{(m)}=y^{(m)}(x_k)$, (m=0,…,8), k=0,…,N-1, for $z_j=j/3$ (j=1,2,3), we obtain the local truncation error formula:

$$\bar{\tau}_k=\mathrm{M}\,\bar{C}_k+\bar{\Psi}_k, \qquad (3.1)$$

where

$$\bar{\Psi}_k=\begin{bmatrix}\sum_{i=0}^{8}\frac{(z_1h)^i}{i!}S_k^{(i)}-y(x_k+z_1h)\\ \sum_{i=0}^{8}\frac{(z_2h)^i}{i!}S_k^{(i)}-y(x_k+z_2h)\\ \sum_{i=0}^{8}\frac{h^i}{i!}S_k^{(i)}-y(x_k+h)\end{bmatrix},\quad \mathrm{M}=\begin{bmatrix}\frac{(z_1h)^9}{9!} & \frac{(z_1h)^{10}}{10!} & \frac{(z_1h)^{11}}{11!}\\ \frac{(z_2h)^9}{9!} & \frac{(z_2h)^{10}}{10!} & \frac{(z_2h)^{11}}{11!}\\ \frac{h^9}{9!} & \frac{h^{10}}{10!} & \frac{h^{11}}{11!}\end{bmatrix},\quad \bar{C}_k=\begin{bmatrix}C_{k,1}\\ C_{k,2}\\ C_{k,3}\end{bmatrix}$$

On the other hand, from the system (2.21), we get

$$\bar{C}_k=A^{-1}\hat{F}_k-A^{-1}\hat{S}_k \qquad (3.2)$$

where A^{-1} is the matrix (2.20), and $\hat{F}_k=[y^{(8)}(x_{k+z_1})\quad ,y^{(8)}(x_{k+z_2})\quad ,y^{(8)}(x_{k+1})]^T$.

Using Taylor's expansions for the functions $y^{(m)}(x), m=0,...,8$ about x_k, in the relation (3.2) and substituting into (3.1), we get the local truncation error at the kth step as follows:

$$\bar{\tau}_k=\mathrm{M}\,(A^{-1}\hat{F}_k+A^{-1}\hat{S}_k)+\bar{\Psi}_k=\begin{bmatrix}\frac{113}{3188646*11!}h^{12}y^{(12)}(x_k)\\ \frac{19456}{1594323*11!}h^{12}y^{(12)}(x_k)\\ \frac{17}{54*11!}h^{12}y^{(12)}(x_k)\end{bmatrix}\quad ,k=0,1,..,N \qquad (3.3)$$

where

$$y(x)=\sum_{i=0}^{12}\frac{(x-x_k)^i}{i!}y^{(i)}(x_k)+O(h^{13}),\ x\in[x_k,x_{k+1}]\ .$$

Note from the relation (3.3) that the local truncation error the presented Spline collocation method is $\|\bar{\tau}_k\|_\infty=\frac{17}{54*11!}y^{(12)}(x_k)h^{12}\equiv O(h^{12})$ and thus the global error after N steps will be $\|T\|_\infty=N.O(h^{12})=\frac{b-a}{h}.O(h^{12})\equiv O(h^{11})$.

Consequently, we have obtained the following: let $y\in C^{11}[a,b]$ be Lipschitz continuous, then the spline approximation $S(x)$ converges to the solution y(x) of the eighth-order BVP as $h\to 0$ for $z_j=j/3$ (j=1,2,3) and

$$\lim_{h\to 0} S^{(m)}(e) = y^{(m)}(e),\ m = 0,...,8,\quad e = a,\ b\ . \qquad (3.4)$$

Furthermore, the convergence order is fourth, i.e., we have

$$|\,y^{(m)}(x) - S^{(m)}(x)\,| < C_m h^{11-m},\ m = 0,...,7\ . \qquad (3.5)$$

Rate of Convergence: Here, the order of convergence is computed when the Spline collocation method applied to linear BVP (3.1) in the interval [0,1]. To do this, without loss of generality, we will assume that $q_0(x) = -1$, $q_i(x) = 0$ $(i = 1,...,7)$, g(x)=0 and initial conditions $y^{(i)}(0) = 1, i = 0,..,7$, with $h = 1/N$. The nodal difference error ε_k^N, is defined by:

$$\varepsilon_k^N = |S_k^N - S_{2k}^{2N}|,\quad k=1,\ldots,N$$

where S_k^N is the spline solution at x_k by the present spline method. The experimental nodal rate of convergence is given by $Rate = Log_2(\varepsilon_k^N / \varepsilon_{2k}^{2N})$.

Table 1 shows spline solutions of test problem in the interval [0,1], for ***N***=10, 20, 40 by presented spline method. The order of convergence for the proposed spline method is computed in **Table 2**.

Table 1: ***The local errors for test problem by presented spline method for N*=10, *h*=0.1**

k	S_k^N	S_{2k}^{2N}	S_{4k}^{4N}
1	1.105170918075647620	1.105170918075647620	1.105170918075647630
2	1.221402758160169320	1.221402758160169820	1.221402758160169830
3	1.349858807575998330	1.349858807576002980	1.349858807576003100
4	1.491824697641250510	1.491824697641269880	1.491824697641270310
5	1.648721270700071010	1.648721270700126980	1.648721270700128130
6	1.822118800390375450	1.822118800390506370	1.822118800390508930
7	2.013752707470204750	2.013752707470471380	2.013752707470476430
8	2.225540928491965920	2.225540928492458320	2.225540928492467450
9	2.459603111156088650	2.459603111156933990	2.459603111156949400
10	2.718281828457648710	2.718281828459020150	2.718281828459044820

Table 2: ***The rate of convergence for presented spline method , with N*=10.**

k	$\varepsilon_k^N = \|S_k^N - S_{2k}^{2N}\|$	$\varepsilon_{2k}^{2N} = \|S_{2k}^{2N} - S_{4k}^{4N}\|$	Rate of Convergence
1	6.50 E-17	1.00 E-18	6.02237
2	5.00 E-16	1.00 E-17	5.64386
3	4.62 E-15	1.01 E-16	5.51547
4	1.909 E-14	4.20 E-16	5.50628
5	5.458 E-14	1.10 E-15	5.6328
6	1.2571 E-13	2.380 E-15	5.72299
7	2.5077 E-13	4.50 E-15	5.8003
8	4.5066 E-13	7.64 E-15	5.88232
9	7.4711E-13	1.187E-14	5.97593
10	1.15979E-12	1.694E-14	6.09729

Notice: the results in the Table 2 show that the rate of convergence for presented spline method bigger than five.

4. Numerical Results and Discussion

The experiments below are designed to test the efficiency of the spline method when applied to linear and nonlinear eighth-order BVPs for two cases of boundary conditions with uniform grids. These problems have exact solutions, thus we compute their actual errors. In calculations, the notations $\delta^{(k)} = \max \| y^{(k)}(x) - S^{(k)}(x) \|$ are used to denote maximum absolute errors, where , k=0,1,..,7 indicate orders of derivatives. Here, all computations were carried out in double precision.

Problem 1. Consider the eighth-order linear BVP (cf. [1,2,4,9,10]):

$$y^{(8)}(x) + x\,y(x) = -(48 + 15x + x^3)\exp(x), \quad 0 \le x \le 1, \qquad (4.1)$$

with two cases of boundary conditions:

Type I:
$$\begin{cases} y(0) = 0,\ y'(0) = 1, y''(0) = 0, y^{(3)}(0) = -3, \\ y(1) = 0,\ y'(1) = -e, y''(1) = -4e, y^{(3)}(1) = -9e \end{cases}, \qquad (4.1a)$$

Type II:
$$\begin{cases} y(0) = 0,\ y''(0) = 0, y^{(4)}(0) = -8, y^{(6)}(0) = -24, \\ y(1) = 0, y''(1) = -4e, y^{(4)}(1) = -16e, y^{(6)}(1) = -36e, \end{cases}. \qquad (4.1b)$$

The exact solution is $y(x) = x(1-x)e^x$. **Table 3** shows comparisons of presented spline method with non-polynomial spline method [**2**], quintic B-spline collocation method [4] and wavelets method [**10**]. In **Table 4**, the absolute errors of problem (4.1)-(4.1b) by presented spline method is compared with non-polynomial spline method [**1**]. **Figures 1-8**, exhibit comparisons of the exact solution $y(x)$ and its derivatives up to seventh-order with the spline solutions obtained by presented spline method.

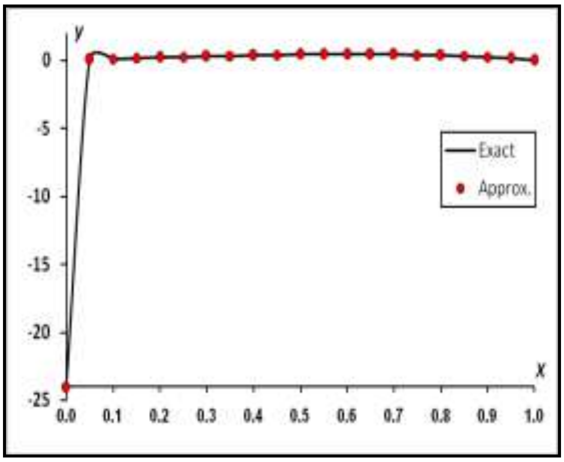

***Fig.*1: *The spline solution S*(*x*) *and the exact solution y*(*x*), *for N=20*.**

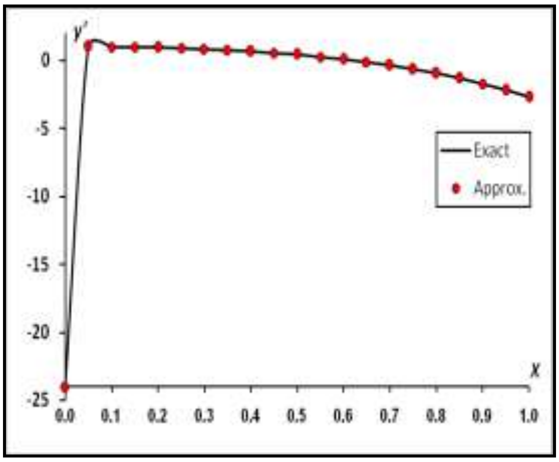

***Fig.2: The spline solution S'*(*x*) *and the exact solution y'*(*x*), *for N=20*.**

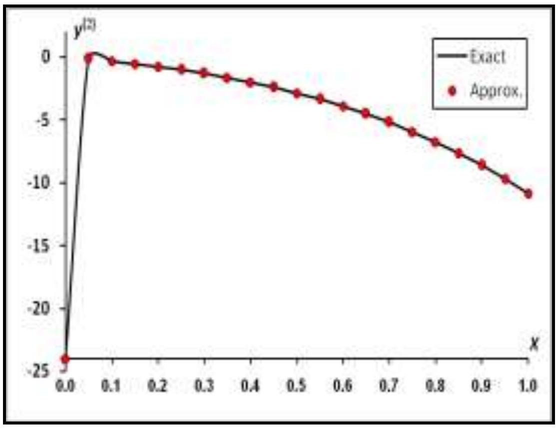

Fig.3: The spline solution $S''(x)$ and the exact solution $y''(x)$, for N=20.

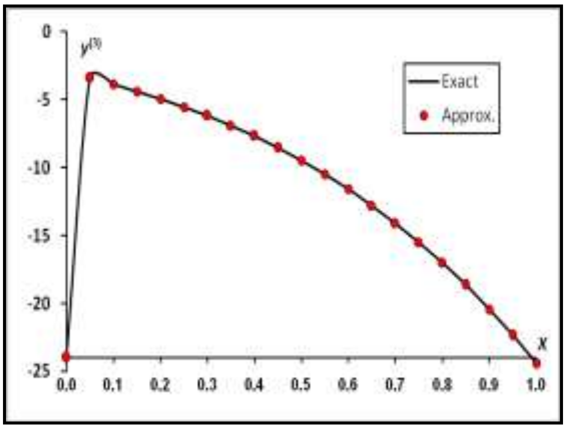

Fig.4: The spline solution $S^{(3)}(x)$ and the exact solution $y^{(3)}(x)$, for N=20.

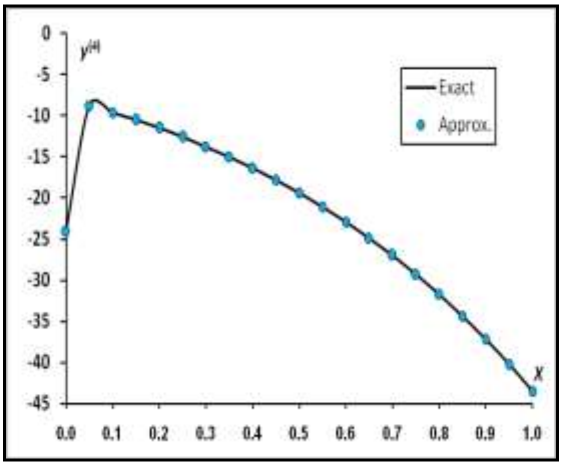

Fig.5: The spline solution $S^{(4)}(x)$ and the exact solution $y^{(4)}(x)$, for N=20.

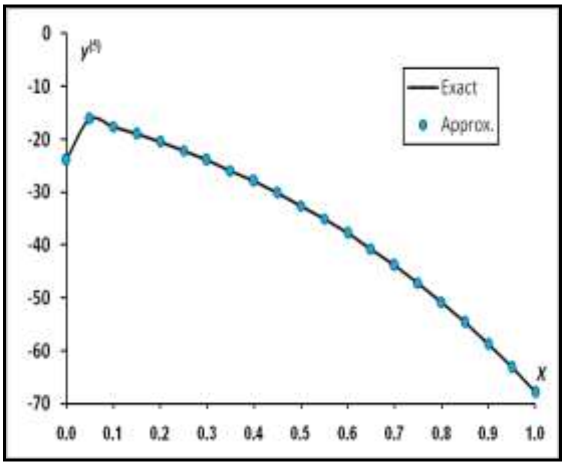

Fig.6: The spline solution $S^{(5)}(x)$ and the exact solution $y^{(5)}(x)$, for N=20.

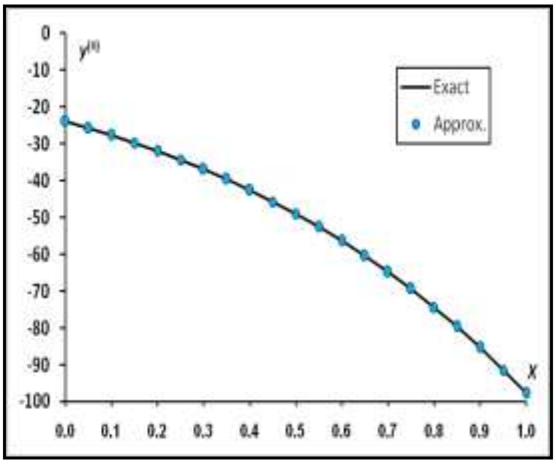

Fig.7: The spline solution $S^{(6)}(x)$ and the exact solution $y^{(6)}(x)$, for N=20.

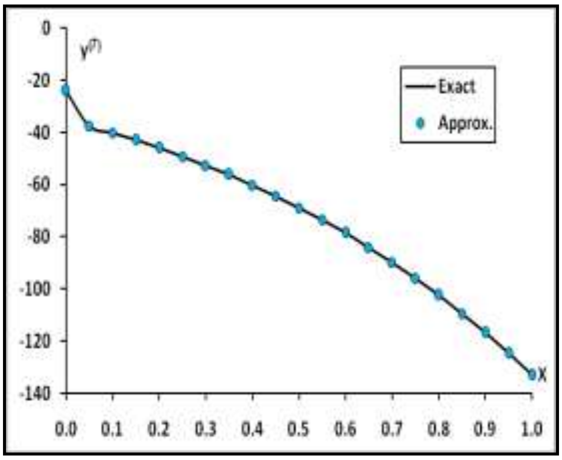

Fig.8: The spline solution $S^{(7)}(x)$ and the exact solution $y^{(7)}(x)$, for N=20.

Table 3: The absolute errors of problem 1, with condition case I.

x_i	Quntic B-Spline Method [4]	Non-poly. Spline Method[2]		Wavelets Method [10] 2M=128	Presented Spline Method
		Fourth-order	Sixth-order		h=1/10
0.1	2.458692E-07	6.20E−13	6.78 E−14	5.9025E −13	2.1705E−15
0.2	8.195639E-07	6.50 E−12	1.46 E−13	5.9733E −12	3.0841E−14
0.3	1.996756E-06	2.00 E−11	5.88 E−13	1.7961E −11	1.4093E−13
0.4	4.291534E-06	3.49 E−11	9.90 E−13	3.1053E −11	3.2936E−13
0.5	6.198883E-06	4.13 E−11	1.07 E−12	3.7057E −11	5.1576E−13
0.6	7.182360E-06	3.49 E−11	8.21 E−12	3.1905E −11	5.9208E−13
0.7	7.033348E-06	2.01 E−11	4.31 E−13	1.8962E −11	4.9751E−13
0.8	5.066395E-06	6.58 E−12	1.30 E−13	6.4797E −12	2.7495E−13
0.9	2.413988E-06	6.37 E−13	1.17 E−14	6.5917E −13	6.5857E−14
1	--------	-------	--------	2.3315E −15	9.9696E−18

Table 4: The absolute errors of Problem 1, with condition case II.

x_i	Non-poly. Spline Method[1]		Presented Spline Method	
	Sixth-order	Tenth-order	h=1/10	h=1/20
0.1	4.31 E−11	6.285E−11	1.3024E−11	2.2944 E−13
0.2	8.20 E−11	1.195E−10	2.4803E−11	4.3686 E−13
0.3	1.12 E−10	1.644E−10	3.4268E−11	6.0348 E−13
0.4	1.32 E−10	1.932E−10	4.0571 E−11	7.1444 E−13
0.5	1.39 E−10	2.031E−10	4.3111 E−11	7.5918 E−13
0.6	1.32 E−10	1.931E−10	4.1561 E−11	7.3198 E−13
0.7	1.12 E−10	1.642E−10	3.5909 E−11	6.3258 E−13
0.8	8.19 E−11	1.193E−10	2.6503 E−11	4.6708 E−13
0.9	4.30 E−11	6.274E−11	1.4113 E−11	2.4890 E−13
1	-------	--------	1.5670 E−20	4.2775 E−20

Problem 2. . Consider the eighth-order linear BVP (cf. [5, 7, 8, 11]):

$$y^{(8)}(x) - y(x) = -8\exp(x), \quad 0 \le x \le 1, \qquad (4.2)$$

with two cases of boundary conditions:

Type I:
$$\begin{cases} y(0)=1,\ y'(0)=0,\ y''(0)=-1,\ y^{(3)}(0)=-2, \\ y(1)=0,\ y'(1)=-e,\ y''(1)=-2e,\ y^{(3)}(1)=-3e, \end{cases} \qquad (4.2a)$$

Type II:
$$\begin{cases} y(0)=1,\ y''(0)=-1,\ y^{(4)}(0)=-3,\ y^{(6)}(0)=-5, \\ y(1)=0,\ y''(1)=-2e,\ y^{(4)}(1)=-4e,\ y^{(6)}(1)=-6e, \end{cases} \qquad (4.2b)$$

Type III: $\begin{cases} y(0)=1,\ y'(0)=0, y''(0)=-1, y^{(3)}(0)=-2, \\ y^{(4)}(0)=-3, y^{(5)}(0)=-4, y^{(6)}(0)=-6, y^{(7)}(0)=-7. \end{cases}$ (4.2c)

Its exact solution is $y(x)=(1-x)\exp(x)$. In **Table5**, maximum absolute errors are calculated by presented spline method for N=10. **Tables 6-7** show comparisons between the absolute errors obtained by presented spline method and other methods in [5,7,8,11].

Table 5: Maximum abs errors of Problem 2 with condition case I, for *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$
5.9E-14	2.5E-13	2.4E-12	4.7E-11	1.3E-09	7.3E-09	2.6E-08	4.6E-08

Table 6: The absolute errors of Problem 2, with condition case II.

x_i	Variational Method [8]	Homotopy Method[11]	Variational Method[7]	Decomposition Method[5]	Presented Spline Method, h=1/10
0.10	-------	2.55072E-09	-------	6.71E-08	1.7439E-0016
0.25	3.8922E−10	------	4.7350E−15	-------	2.7051E-0015
0.40	-------	3.39829E-09	-------	2.06E-07	1.3307E-0014
0.50	1.1571E−07	-------	8.2507E−13	-------	3.2017E-0014
0.60	-------	3.93756E-09	-------	2.08E-07	5.0898E-0014
0.75	1.0479 E−06	-------	1.5260E−11	-------	5.8968E-0014
0.80	-------	4.45065E-09	-------	1.29E-07	2.7675E-0014
1.00	4.2188E−06	4.93598E−09	1.1278E−10	0.000000	2.4384E-0018

Table 7: The absolute errors of Problem 2, with condition case III.

x_i	Homotopy Method[11]	Presented Spline Method h=1/10
0.1	1.11022 E-16	1.0842E-19
0.2	1.11022 E-16	3.4152E-18
0.3	3.33067 E-16	2.0220E-17
0.4	1.11022 E-15	6.7925E-17
0.5	1.11022 E-16	1.6821E-16
0.6	2.22045 E-16	3.3724E-16
0.7	1.11022 E-16	5.6693E-16
0.8	1.60982 E-15	7.9152E-16
0.9	7.49401 E-16	8.3425E-16
1.0	0.0	3.2725E-16

Problem 3. We consider the following nonlinear BVP (cf. [11]):

$$\begin{cases} y^{(8)}=\exp(-x)\, y^{(2)}(x), \quad 0\le x\le 1, \\ y(0)=y'(0)=y''(0)=\cdots=y^{(7)}(0)=1. \end{cases}$$

The exact solution is $y(x)=\exp(x)$. **Table 8** appears comparisons of the numerical solution and absolute errors by presented spline method with other by the Homotopy method [**11**]. In **Table9**, maximum absolute errors are computed by presented spline method for N=10.

Table 8: The numerical solution and abs errors of nonlinear problem 3, for *h*=0.1.

x_i	*Homotopy Method* [11, 2010]			Presented Spline Methods	
	Exact solution	Homo. Sol.	Abs Error	Spline Sol.	Abs Error
0.1	1.1051709180756	1.10517	1.45978 E-07	1.1051709180756	1.08420 E-19
0.2	1.2214027581602	1.2214	1.02754 E-07	1.2214027581602	5.13153E-16
0.3	1.3498588075760	1.34986	2.04184 E-07	1.3498588075760	4.77320E-15
0.4	1.4918246976413	1.49182	1.49324 E-07	1.4918246976413	1.98108E-14
0.5	1.6487212707001	1.64872	1.06158 E-07	1.6487212707001	5.71415E-14
0.6	1.8221188003905	1.82212	1.44092 E-07	1.8221188003904	1.33523E-13
0.7	2.0137527074705	2.01375	1.05881 E-07	2.0137527074702	2.71775E-13
0.8	2.2255409284925	2.22554	1.4508 E-07	2.2255409284920	5.01690E-13
0.9	2.4596031111569	2.4596	1.62443 E-07	2.4596031111561	8.61010E-13
1	2.7182818284590	2.71828	1.65095 E-07	2.7182818284576	1.39653E-12

Table 9: Maximum abs errors of Problem 3, for *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$	$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$
1.4E-12	6.4E-12	2.3E-11	6.6E-11	4.5E-11	1.4E-11	6.4E-12	6.8E-13

Problem 4. Consider the following general eighth-order BVP:

$$y^{(8)} + y^{(7)} + y^{(6)} + y^{(5)} + y^{(4)} + y^{(3)} + y'' + y' + y = \cos(x) + \sin(x),\ x \in [-\pi/2, \pi/2],$$

$$y(-\pi/2) = -1,\quad y'(-\pi/2) = 1,\ \ y''(-\pi/2) = 1, y^{(3)}(-\pi/2) = -1,$$

$$y(\pi/2) = 1,\quad y'(\pi/2) = -1,\ \ y''(\pi/2) = -1,\ y^{(3)}(\pi/2) = 1$$

The exact solution is $y(x) = \cos(x) + \sin(x)$. In **Table10,** the absolute errors are calculated by proposed spline method,. The spline solutions $S(x)$,$S'''(x)$, $S^{(6)}(x)$ as well as the solutions $y(x)$, ,$y'''(x)$, $y^{(6)}(x)$ are illustrated in **Figs.9-14,** respectively.

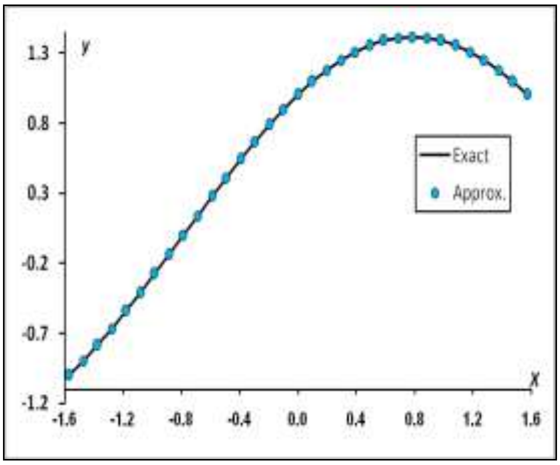

***Fig*.9: *The spline solution S*(*x*) *and the exact solution y*(*x*), *for Problem4, h*=Pi/32.**

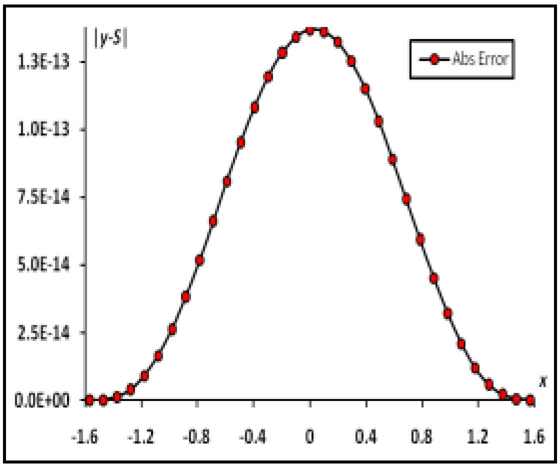

***Fig*.10: *The absolute error in spline solution S*(*x*) , for *N*=32.**

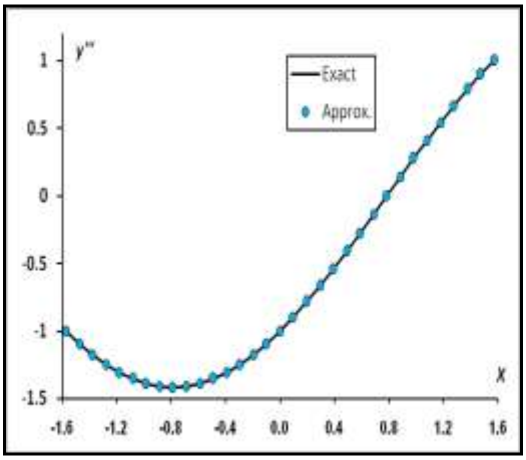

Fig.11: *The spline solution S'''*(*x*) *and the exact solution y'''*(*x*), *for Problem4, h*=Pi/32.

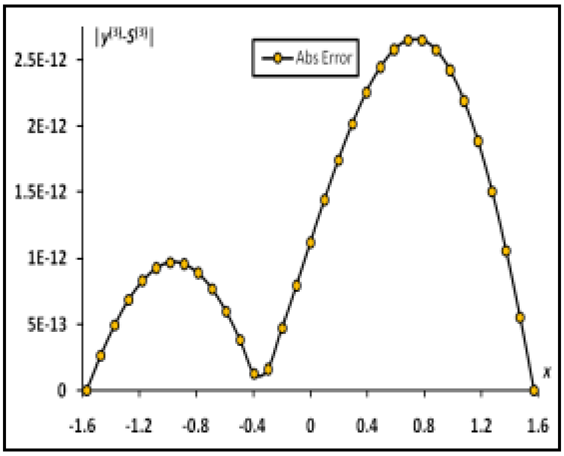

Fig.12: *The absolute error in spline solution S'''*(*x*) , for *h*=Pi/32.

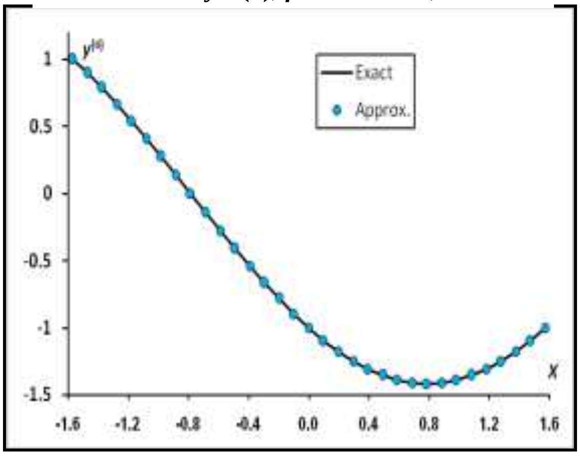

Fig.13: *The spline solution* $S^{(6)}(x)$ *and the exact solution* $y^{(6)}(x)$, *for Problem4, h*=Pi/32.

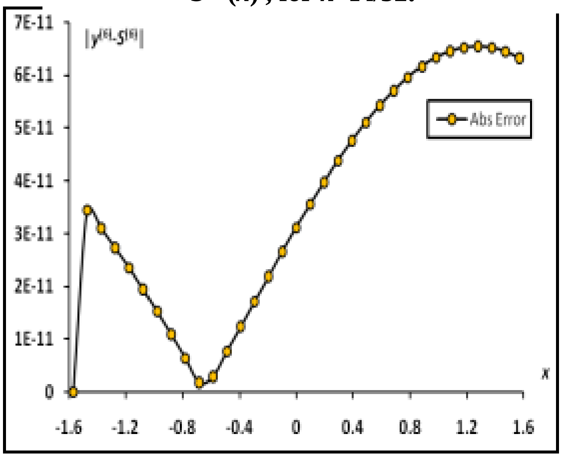

Fig.14: *The absolute error in spline solution* $S^{(6)}(x)$, for *h*=Pi/32.

Table 10: The absolute errors of Problem 4, for *N*=32 by the presented spline method.

x_i	$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$
$-\frac{3\pi}{8}$	8.951E-15	6.368E-14	2.705E-13	8.318E-13	1.677E-11
$-\frac{\pi}{4}$	5.171E-14	1.436E-13	1.042E-13	8.905E-13	4.702E-12
$-\frac{\pi}{8}$	1.083E-13	1.249E-13	2.025E-13	1.271E-13	8.328E-12
0	1.369E-13	1.033E-14	3.694E-13	1.121E-12	2.093E-11
$\frac{\pi}{8}$	1.151E-13	1.136E-13	2.636E-13	2.258E-12	3.152E-11
$\frac{\pi}{4}$	5.942E-14	1.504E-13	3.125E-14	2.652E-12	3.841E-11
$\frac{3\pi}{8}$	1.206E-14	7.698E-14	2.375E-13	1.888E-12	4.008E-11

$\frac{\pi}{2}$	1.214E-17	1.881E-17	2.288E-17	1.821E-17	3.538E-11

x_i	$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$	$\delta^{(8)}$
$-\frac{3\pi}{8}$	6.367E-11	2.349E-11	3.810E-11	3.277E-11
$-\frac{\pi}{4}$	7.058E-11	6.399E-12	5.140E-11	2.146E-11
$-\frac{\pi}{8}$	6.848E-11	1.240E-11	6.005E-11	4.447E-12
0	5.717E-11	3.112E-11	6.177E-11	1.568E-11
$\frac{\pi}{8}$	3.753E-11	4.758E-11	5.535E-11	3.588E-11
$\frac{\pi}{4}$	1.150E-11	5.959E-11	4.079E-11	5.306E-11
$\frac{3\pi}{8}$	1.805E-11	6.524E-11	1.935E-11	6.463E-11
$\frac{\pi}{2}$	4.762E-11	6.322E-11	6.662E-12	6.880E-11

5. Conclusion

Spline collocation method is successfully applied with three collocation points for the numerical solutions of linear and nonlinear eighth-order boundary value problems with two cases of boundary conditions. The presented spline method is tested on four problems. Comparisons of the results obtained by the present spline method with obtained by non-polynomial spline methods [1,2008]-[2,2009], Quintic B-spline collocation method [4,2012], modified decomposition method [5,2007], variational methods [7,2008]-[8,2010] wavelets method [10, 2010] and homotopy method [11, 2010] reveal that the present method is very effective and convenient.

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