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The Boundary Propertieses of Some Functionals in Class C_{α}^{β}

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□ ABSTRACT □

This paper presents a certain method to determine the range of variability (or the set of values) of some functionals defined in the Class C_{α}^{β} (i.e the class of analytic functions in the unit disk $D(|z|<1)$ of the form:

$$f(z)=\int_{-\pi}^{+\pi}\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\,d\mu(t)\,, \qquad f(0)=1\,,\ 0<\alpha,\beta<1$$

It have been shown in this class that the range of variability of the functional $F(f)=f(z_0)$ is the closed disk $|w-w_0|\le R$ where

$$w_0=\frac{1+\alpha\beta r^2}{1-\alpha^2 r^2}\,,\quad R=\frac{(\beta+\alpha)r}{1-\alpha^2 r^2}\,,\quad |z_0|=r\,,\ 0<r<1$$

The estimations of modulus of function and some other estimations related were also obtaind.

Key Words:
Convex Hull
Range of variability
Functional

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الخواص الحدية لبعض الداليات العقدية في الفضاء C_α^β

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☐ ملخّص ☐

يقدم هذا البحث طريقة معينة لتحديد مستقرات بعض الداليات العقدية المختارة في الفضاء C_α^β وهو فضاء التوابع التحليلية في قرص الواحدة التي تقبل التمثيل التكاملي الآتي:

$$f(z)=\int_{-\pi}^{+\pi}\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\,d\mu(t)\ , \qquad f(0)=1,\ 0<\alpha,\beta<1$$

وقد تم البرهان على أن مستقر الدالي $F(f)=f(z_0)$ في هذا الفضاء هو القرص المغلق $|w-w_0|\le R$ حيث

$$w_0=\frac{1+\alpha\beta r^2}{1-\alpha^2r^2}\ ,\quad R=\frac{(\beta+\alpha)r}{1-\alpha^2 r^2}\ ,\quad |z_0|=r\ ,\quad 0<r<1$$

كما تم الحصول على تقدير طويلة التابع في هذا الفضاء وتقديرات أخرى مرتبطة به .

الكلمات المفتاحية:
الغلاف المحدب
المستقر
الدالي

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Introduction

One of the fundamental extremal problems considered in the domain of complex functions is concerned with determining the range of variability (or the set of values) of the functional

$$(1) \qquad J(f) = F[f(z_0), f'(z_0), \dots, f^{(n)}(z_0)]$$

defined on some space E of analytic functions where z_0 is a fixed point of the domain in which the function f is defined. Denote this range by B and his boundary by Γ. If the space E is compact and connected then B is closed and also connected [5]. So, in order to characterize B it is enough to determine his boundary Γ.

A survey of methods and results can be found among others in [2], [6] and [10].

In this paper we shall present a certain method to determine the range of variability of the functional (1) defined in the class of functions f possessing an integral representation in the unit disk $D(|z| < 1)$ when it has the linear form:

$$(2) \qquad J(f) = F\left(f, f', \dots, f^{(n)}\right) = \sum_{k=0}^{n} a_k(z_0)\; f^{(k)}(z_0)$$

where $a_k(z_0),\; k = 0,1,\dots n$ are some complex functions in D

The aims and importance

It is justified by various reasons to investigate the integral functionals in spaces of complex functions because the fundamental functionals still very often examined are expressed in a simple way by integrals.

Methods

We shall apply the method of Structural Formulas [1] for classes of functions having an integral representation in Stieltjes sense like E_q and C_α^β (which would be soon defined) and use other properties like convexity and connectedness of these spaces.The virtue of this method is that one can obtain good results using simple tools.

Consideration and Results in E_q

Let E_q denote the class of functions f given by the formula

$$(3) \qquad f(z) = \int_a^b q(z,t)\,d\mu(t)$$

where $q(z,t)$ is an analytic function in the unit disk D for every fixed $t \in [a,b]$ and $\mu(t)$ is a nondecreasing function in the interval $[a,b]$ such that $\mu(b) - \mu(a) = 1$. Denote the class of functions $\mu(t)$ by $U[a,b]$

This class is known as the class of functions possessing an integral representation and (3) is known as the Structural Formula for the class E_q. In this class the following properties are true [1]:

Property 1. The class E_q is compact and connected in the topology of almost uniform convergence.

Property 2. If $f \in E_q$ then the set B of values of the functional

$$(4) \qquad J(f) = f(z_0) = \int_a^b q(z_0, t) d\mu(t), \quad \mu \in U[a,b]$$

is closed, connected and convex and its boundary is a curve given by :

$$(5) \qquad \Gamma : w(t) = q(z_0, t), \quad a \le t \le b$$

The convexity of $F(f)$ here follows from the convexity of μ .

An essential example of this class is the Caratheodory Class [9], i.e. the class of analytic functions f in the unit disk D with a positive real part and $f(0) = 1$ (this class is denoted by C). It is well known ([3]and [8]) that the functions of this class has the following integral representation:

$$(6) \qquad f(z) = \int_{-\pi}^{+\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

where $\mu \in U[-\pi, \pi]$ and $\mu(+\pi) - \mu(-\pi) = 1$).

We see that the Class C coincides with the class E_q when we put

$$q(z,t) = \frac{e^{it} + z}{e^{it} - z_0}, \quad a = -\pi, \quad b = \pi .$$

Another example is the Class of analytic functions of the form:

$$(7) \qquad f(z) = \int_{-\pi}^{+\pi} \frac{1 + e^{it} z}{1 - \alpha e^{it} z} d\mu(t) \quad f(0) = 1, \ -1 \le \alpha \le 1$$

defined in the unit disk with $\mu \in U[-\pi, \pi]$. This class was denoted by C_α and studied in [4].

We now return the functional (2) and show the following theorem: .

Theorem 1. If $a_k(z), k = 0,1,...n$ are some continuous functions on the unit disk D and z_0 is some fixed point in this disk, then the set B of values of the functional

$$(8) \qquad J(f)=\sum_{k=0}^{n} a_k(z_0)\, f^{(k)}(z_0)\,, \quad f\in E_q$$

is closed , convex and bounded by the curve Γ, which equation is:

$$(9) \qquad \Gamma : W(t)=W(z_0,t)\,, \quad a\le t\le b$$

where

$$W(z_0,t)=\sum_{k=0}^{n} a_k(z_0) q_z^{(k)}(z_0,t)$$

and $q(z_0,t)$ is given by (3).

Proof. Differentation of both sides in the formula (3) gives the relations:

$$f^{(k)}(z)=\int_a^b q_z^{(k)}(z,t)d\mu(t) \quad k=0,1,2,3,...,$$

with $f^{(0)}(z)=f(z)$. And so:

$$J(f)=\sum_{k=0}^{n} a_k(z)\int_a^b q_z^{(k)}(z,t)d\mu(t)=\int_a^b \sum_{k=0}^{n} a_k(z) q_z^{(k)}(z,t)d\mu(t).$$

Puttting

$$(10) \qquad W(z_0,t)=\sum_{k=0}^{n} a_k(z_0) q^{(k)}(z_0,t)$$

we obtain the functional (8) depending only on $\mu(t)$ and so we can write:

$$(11) \qquad J(f)=\Phi(\mu)=\int_a^b W(z_0,t)d\mu(t).$$

Noticing the analogy between $W(z_0,t)$ in (11) and $q(z_0,t)$ in (4) it follows immediately from property2 that the set B is closed and convex with boundary Γ given by:

$\Gamma : w(t)=W(z_0,t), \quad a\le t\le b$

where $W(z_0,t)$ is given by (10).

The problem in the Class C_α^β

Let C_α^β denote the class of functions given by the formula:

$$(12) \qquad f(z)=\int_{-\pi}^{+\pi} \frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\, d\mu(t)$$

where $z \in D$, $0<\alpha<1$, $0<\beta<1$ and $\mu \in U[-\pi,\pi]$. Note that the Class C_α^β coincides with the family E_q when we put :

$$(13) \qquad q(z,t)=\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\ , \quad a=-\pi,\ b=\pi .$$

Theorem 2 . The set of values of the functional (2) with $f \in C_\alpha^\beta$ is closed, convex and bounded by curve Γ which equation is:

$$(14) \quad H(z_0,t)=a_0(z_0)\frac{1+\beta e^{it}z_0}{1-\alpha e^{it}z_0}+\sum_{k=1}^{n}a_k(z_0)\frac{n!(\beta+\alpha)\alpha^{k-1}e^{ikt}}{(1-\alpha e^{it}z_0)^{k+1}}\ ,\ -\pi\le t\le\pi .$$

Proof. Differentating both sides in the formula (12) we obtain :

$$f^{(0)}(z)=\int_{-\pi}^{+\pi}\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\,d\mu(t)$$

$$f'(z)=\int_{-\pi}^{+\pi}\frac{(\beta+\alpha)e^{it}}{(1-\alpha e^{it}z)^2}\,d\mu(t),\ \ldots.,$$

$$f^{(n)}(z)=\int_{-\pi}^{+\pi}\frac{n!(\beta+\alpha)\alpha^{n-1}e^{int}}{(1-\alpha e^{it}z)^{n+1}}\,d\mu(t),$$

and noticing 1n (13) that

$$q^{(0)}(z_0,t)=\frac{1+\beta e^{it}z_0}{1-\alpha e^{it}z_0}\ ,\ \ldots\ldots,\ q^{(n)}(z_0,t)=\frac{n!(\beta+\alpha)e^{int}}{(1-\alpha e^{it}z_0)^{n+1}}$$

we get:

$$J(f)=\sum_{k=0}^{n}a_k(z_0)\ f^{(k)}(z_0)=$$

$$=a_0\int_{-\pi}^{+\pi}\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}\,d\mu(t)+\sum_{k=0}^{n}a_k(z_0)\int_{-\pi}^{+\pi}\frac{k!(\beta+\alpha)\alpha^{k-1}e^{ikt}}{(1-\alpha e^{it}z)^{k+1}}\,d\mu(t)\ =$$

$$=\int_{-\pi}^{+\pi}[a_0\frac{1+\beta e^{it}z}{1-\alpha e^{it}z}+\sum_{k=0}^{n}a_k(z_0)\frac{k!(\beta+\alpha)\alpha^{k-1}e^{ikt}}{(1-\alpha e^{it}z)^{k+1}}]d\mu(t)\ .$$

and so we get (14) immediately from theorem1.

Notice that heorem2 (and other properties) remain true if we assume in definition that $|\alpha|\le 1$, and $|\beta|\le 1$

Example. Find the sets of values of the functional

$$F(f)=f(z_0)+\frac{1}{2}f'(z_0)+\frac{1}{4}f''(z_0)$$

at point $z_0=0$ for $f\in C_\alpha^\beta$ and $\alpha=\beta=1$.

Solution: Making use of (14) we have at point $z_0=0$:

$$h(t)=H(0,t)=1+\frac{1}{2}(\beta+\alpha)e^{ikt}+\frac{1}{4}2(\beta+\alpha)\alpha e^{i2t}=1+e^{it}+e^{2it} \quad .$$

So by theorem 2 the set B of values of the given functional is bounded by the curve Γ given by equation $\Gamma:\ h=h(t)$ where

$$h(t)=1+e^{it}+e^{2it}, \quad -\pi\le t\le\pi \ .$$

To find the form of this curve we put $h(t)$ in the form $h=u+iv$ so that:

$$u=1+cost+cos2t$$

$$v=sint+sin2t$$

and

$$(u-1)^2+v^2=(cost+cos2t)^2+(sint+sin2t)^2=2+2cost$$

On the other hand by putting:

$$u=cost(1+2cost)$$

$$v=sint(1+2cost)$$

we can omit the parameter t by squaring and adding both sides after taking into account that

$$1+2cost=(u-1)^2+v^2-1 .$$

In this case we get the equation of Γ in the following Cartesian form:

$$\left((u-1)^2+v^2-1\right)^2=u^2+v^2$$

and this at the same time determines the boundary of B (the set of values of given functional) .

The functional $f(z_0)$

Theorem 3. The set of values of the functional $F(f)=f(z_0)$ when $f\in C_\alpha^\beta$ coincides with the closed disk $|w-w_0|\le R$ where

$$(15) \qquad w_0=\frac{1+\alpha\beta r^2}{1-\alpha^2r^2}, \quad R=\frac{(\beta+\alpha)r}{1-\alpha^2 r^2}, \ |z_0|=r, \ 0<r<1$$

Proof. Let B be the set of values of the functional

$F(f)=f(z_0), \quad f\in C_\alpha^\beta$.

According to theorem 2 the set B is closed and convex and its boundary is given by equation $\Gamma:\ w=w(t)$ where

$$(16) \qquad w(t)=\frac{1+\beta e^{it}z_0}{1-\alpha e^{it}z_0}, \ -\pi\le t\le\pi .$$

To find the form of this curve we put $w(t)$ in the form $w(t)=u+iv$ and assume that $z_0=x_0+y_0=re^{i\theta}$. Then we get

$$u=\frac{1-\alpha\beta r^2+(\beta-\alpha)(x_0\cos\varphi+y_0\sin\varphi)}{1+\alpha^2r^2-2\alpha(x_0\cos\varphi+y_0\sin\varphi)}$$
$$v=\frac{(\beta+\alpha)(x_0\sin\varphi-y_0\cos\varphi)}{1+\alpha^2r^2-2\alpha(x_0\cos\varphi+y_0\sin\varphi)}$$

Where $|z_0|=r$ and $\varphi=t+\theta$. Now by putting:
$p=(x_0\cos\varphi+y_0\sin\varphi)$,
$q=(x_0\sin\varphi-y_0\cos\varphi)$
we find p and q in the form:

$$p=\frac{u+\alpha^2r^2u+\alpha\beta r^2-1}{2\alpha u+\beta-\alpha}$$
$$q=\frac{v}{\beta+\alpha}\left(1+\alpha^2r^2-2\beta\frac{u+\alpha^2r^2u+\alpha r^2-1}{2\alpha u+\beta-\alpha}\right)$$

Now noticing that $p^2+q^2=|z_0|^2=r^2$ the parameter φ can be omitted by squaring and adding both sides in the above equations . A simple calculation shows that:

$$(17)\qquad u^2+v^2-2\frac{1+\alpha\beta r^2}{1-\alpha^2r^2}u+\frac{1-\beta^2r^2}{1-\alpha^2r^2}=0$$

and this equation leads to the following one:

$$(18)\qquad \left(u-\frac{1+\alpha\beta r^2}{1-\alpha^2r^2}\right)^2+v^2=\left(\frac{(\beta+\alpha)r}{1-\alpha^2r^2}\right)^2.$$

The last equation represents a circle $|w-w_0|=R$ with center and radius given by (15).Thus the closed disk $|w-w_0|\le R$ is the set of values of the given functional.

Some estimations in the Class C_α^β

Theorem 2 helps to get the estimations of modulus of function in the Class C_α^β and some other estimations related.

Theorem 4. In the Class C_α^β the following estimations are true:

$$\frac{1-\beta r}{1+\alpha r} \le |f(z)| \le \frac{1+\beta r}{1-\alpha r} \qquad (19)$$

$$|Im\ f(z)| \le \frac{(\beta+\alpha) r}{1-\alpha^2 r^2} \qquad (20)$$

$$\left|f^{(n)}(z)\right| \le \frac{n!(\beta+\alpha)\alpha^{n-1}}{(1-\alpha r)^{n+1}}\ ,\ n=1,2,3,... \qquad (21)$$

For every $z = re^{i\theta}\ ,\ 0<r<1\ ,\ -\pi \le \theta \le \pi$

Proof. 1) To prove (19) we notice that theorem 2 (relation (18)) gives:

$$\left| |f(z_0)| - \left| \frac{1+\alpha\beta r^2}{1-\alpha^2 r^2} \right| \right| \le \left| f(z_0) - \frac{1+\alpha\beta r^2}{1-\alpha^2 r^2} \right| \le \frac{(\beta+\alpha) r}{1-\alpha^2 r^2}$$

for every fixed point z_0 chosen arbitrarily in the unit disk, with $|z_0| = r$. So the following inequalities are true in the hole unit disk D:

$$|f(z)| \le \frac{1+\alpha\beta r^2}{1-\alpha^2 r^2} + \frac{(\beta+\alpha) r}{1-\alpha^2 r^2} = \frac{1+\beta r}{1-\alpha r}$$

and

$$|f(z)| \ge \frac{1+\alpha\beta r^2}{1-\alpha^2 r^2} - \frac{(\beta+\alpha) r}{1-\alpha^2 r^2} = \frac{1-\beta r}{1+\alpha r}$$

where $|z| = r$, and so (19) is true.

2) To prove (20) its enough to notice in theorem 2 that the center of the disk $|w-w_0| \le R$ lies on the real axis . And then (20) follows from the relations:

$$-R \le Im\ f(z) \le R.$$

3) Finally since the n-th derivative of function $f(z)$ is given by the formula

$$(22) \qquad f^{(n)}(z) = \int_{-\pi}^{+\pi} \frac{n!(\beta+\alpha)\alpha^{n-1} e^{int}}{\left(1-\alpha e^{it} z\right)^{n+1}}\ d\mu(t),\ n = 1,2,3,...$$

then (21) holds by the inequalities:

$$\left|f^{(n)}(z)\right| \le \left| \int_{-\pi}^{+\pi} \frac{n!(\beta+\alpha)\alpha^{n-1} e^{int}}{\left(1-\alpha e^{it} z\right)^{n+1}} d\mu(t) \right| \le \int_{-\pi}^{+\pi} \left| \frac{n!(\beta+\alpha)\alpha^{n-1} e^{int}}{\left(1-\alpha e^{it} z\right)^{n+1}} \right|\ d\mu(t)$$

$$\le \frac{n!(\beta+\alpha)\alpha^{n-1}}{(1-\alpha|z|)^{n+1}} \int_{-\pi}^{+\pi} d\mu(t) = \frac{n!(\beta+\alpha)\alpha^{n-1}}{(1-\alpha|z|)^{n+1}}\ ,\ n=1,2,3,...\ ,$$

where $|z| = r < 1$.

The Estimations of Coefficients

Theorem 5. If $f \in C_\alpha^\beta$ and

$$(23) \qquad f(z) = 1 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots, |z| < 1$$

then the following estimations are true:

$$(24) \qquad |b_n| \le (\beta + \alpha)\alpha^{n-1},\ n = 1,2,3,\ldots$$

Proof. The coefficients of the function $f(z)$ in Taylor Expansion (23) are given by the relations:

$$b_n = \frac{f^{(n)}(0)}{n!}, \qquad n = 1,2,3,\ldots$$

where , by (22)**:**

$$f^{(n)}(0) = \int_{-\pi}^{+\pi} n!(\beta + \alpha)\alpha^{n-1} e^{int}\ d\mu(t) = n!(\beta + \alpha)\alpha^{n-1} \int_{-\pi}^{+\pi} e^{int}\ d\mu(t),$$

for $n = 1,2,3,\ldots,$. Hence according to (21):

$$|b_n| = \left| \frac{f^{(n)}(0)}{n!} \right| \le \frac{n!(\beta + \alpha)\alpha^{n-1}}{n!} = (\beta + \alpha)\alpha^{n-1},\ n = 1,2,3,\ldots$$

and so the inequalities (24) hold.

Remarks and results

If we put $\alpha = \beta = 1$ then the Class C_α^β coincides with the Class C defined by(7). As a consequence of theorems 4 the following properties are true in Class C:

$$(25) \qquad \frac{1 - r}{1 + r} \le |f(z)| \le \frac{1 + r}{1 - r}$$

$$(26) \qquad \frac{1 - r}{1 + r} \le |Re\, f(z)| \le \frac{1 + r}{1 - r}$$

$$(27) \qquad |Im\, f(z)| \le \frac{2r}{1 - 2r^2}$$

$$(28) \qquad \left| f^{(n)}(z) \right| \le \frac{2n!}{(1 - r)^{n+1}},\ n = 1,2,3,\ldots$$

for every $z = re^{i\theta} \in D$**.**

Besides this the coefficients of the function $f(z) \in C$ in Taylor Expansion (23) satisfy the inequalities:

$$|b_n| \le (\beta + \alpha)\alpha^{n-1} = 2,\ n = 1,2,3,\ldots$$

which shows that the estimates $|b_n| \le 2$ hold for every n .

If just $\beta = 1$ then the Class C_α^β coincides with the Class C_α defined by(7) and similar estemations to (19)- (21) and (24)hold [4].

Open problems

1. Determinate the range of variability of functionals

$$(29) \qquad z_0 f(z_0),\ z_0^2 f(z_0),\ z_0 f'(z_0),\ z_0^2 f'(z_0),$$

when $f \in C_\alpha^\beta$ and z_0 is an arbitrary point of the unit disk such that $Re\, z_0 > 0$.

2. Let T denote the family of functions

$$f(z) = z + a_2 z^2 + ... + a_n z^n + ..., \qquad |z| < 1$$

where a_n are real numbers[11] . It is well known [12] that the functions of this class can be represented by the following (structural) formula

$$f(z) = \int_{-\pi}^{+\pi} \frac{z}{1 - 2z\cos t + z^2}\, d\mu(t)$$

where $\mu(t) \in U[-\pi, +\pi]$. The functions of this class belong to E_q. How to determine the range of variability of t functionals (29)**?**

3. In Class T consider similar estimations to (25)- (28)**.**

References:

[1] ALEKSANDROV,I. *Boundary Values of Functional on the Class of Holomorphic Functions Univalent in a Circle*. Sibirsk, Mat. Z. 4 , (1963),17-31.

[2] BABALOLA. T, K. O. OPOOL, O. *Iterated integral Transforms of Caratheodory Functions and their Applications to Analytic and Univalent Functions*. Tamking Journal of Mathemtics Volume 37, Number 4, 355-366, Winter 2006

[3]BADDOUR,H. *About the range of variability of linear functionals in Caratheodory Classe*. Damascus univ.journal- No.28 – 1998

[4]BADDOUR,H.*The Boundary Propertieses of Some Functionals in Clas*C_α . Tichreen University Journal, Bas Sciences Seies2012 .V 34, Nr (1*).*

[5]GOEL , R. *A Class of Close-to-convex Functions. Czchoslovak Mathematical Journal 18 (93) (1968) , 503-508*P.W.N Warsaw 1975 .

[6]JANOWSKI, W. *About Some Family of Univalent Functions*. Annales Polonici Mathematici XVIII (1966) , 171-203

[7] KHAVINSON, D. STESSIN, M. *Certain linear extremal problems in Bergman spaces of analytic functions,* Indiana Univ. Math. J. 46 (1997), no. 3, 933–974.

[8] KRZYZ J. *Theory and Problems in Analytic Functions*. P.W.N Warsaw 1975 .

[9] NUNOKAWA, M. YAVUZ DUMAN, E. *Properties of functions concerned with Caratheodory functions***.** ANNALES ,U M C-S **,** Lublin – Polonia Vol. LXVII, No. 2, 2013 Sectio A 33–41

[10] POMMERENKECh *Univalent Functions*. Vandehhoeck &Go`ttingen 1975.

[11] ROGOSINSKI, W. *Uber positive harmonishe Entwicklungen und typisch-reelle Potenzreihen*, Mat. Z. 35 (1932) p 93-121.

[12] ROGOSINSKI, W. SHAPIRO, H. *On certain extremum problems for analytic functions*, Acta Math. 90 (1953), 287–318.