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Spline Method with Three Collocation Parameters for Solving Ninth-Order General Differential Equations Subject to Boundary Conditions

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□ ABSTRACT □

In this paper, we use polynomial splines of eleventh degree with three collocation points to develop a method for computing approximations to the solution and its derivatives up to ninth order for general linear and nonlinear ninth-order boundary-value problems (BVPs). The study shows that the spline method with three collocation points when is applied to these problems is existent and unique. We prove that the proposed method if applied to ninth-order BVPs is stable and consistent of order eleven, and it possesses convergence rate greater than six.

Finally, some numerical experiments are presented for illustrating the theoretical results and by comparing the results of our method with the other methods, we reveal that the proposed method is better than others.

Keywords: Polynomial Splines, Collocation Points, Linear Ninth-Order BVPs, Error Estimation, Stability, Convergence.

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طريقة شرائحية بثلاثة وسطاء تجميع لحل مسائل في المعادلات التفاضلية المعممة من المرتبة التاسعة خاضعة لشروط حدية.

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□ ملخّص □

يتم في هذا العمل استخدام كثيرات حدود شرائحية من الدرجة الحادية عشرة مع ثلاث نقاط تجميع لتطوير طريقة لحساب الحل العددي ومشتقاته حتى المرتبة التاسعة لمسائل القيم الحدية الخطية وغير الخطية في المعادلات التفاضلية المعممة من المرتبة التاسعة.

تبين الدراسة أن الطريقة الشرائحية المقترحة عندما طُبِقتْ بثلاث نقاط تجميع لهذه المسائل كانت موجودة ومعرفة بشكل وحيد. كما تظهر الدراسة التحليلية للتقارب أن الطريقة المقترحة مستقرة ومتناسقة من الرتبة الحادية عشرة وتملك معدل تقارب يزيد عن ستة. كما تم اختبار الطريقة الشرائحية بحل بعض المسائل التطبيقية، إذ تشير المقارنات لنتائجنا مع نتائج عددية لبعض الطرائق المذكورة في مراجع أخرى حديثة إلى أفضلية النتائج التي توصلنا إليها من حيث الاستقرار والدقة العددية.

الكلمات المفتاحية: كثيرات حدود شرائحية، نقاط تجميع، مسائل القيمة الحدية الخطية من المرتبة التاسعة ،تقدير الخطأ، الاستقرار، التقارب.

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1. Introduction

Higher-order boundary value problems are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, engineering and applied physics [1-15]. The several spline methods have been extensively applied in numerical ordinary differential equations due to its easy implementation and high-order accuracy [6-8,9-11].

We present in this paper, new spline collocation method of the numerical solutions for two types of problems. **The first type**, general linear ninth-order BVPs of the form:

$$y^{(9)}(x)+\sum_{i=0}^{8}q_i(x)\,y^{(i)}(x)=g(x),\quad x\in\,[a,b], \tag{1.1}$$

subject to the following boundary conditions:

$$\begin{cases} y(a)=\alpha_0,\; y'(a)=\alpha_1, y''(a)=\alpha_2, y^{(3)}(a)=\alpha_3, y^{(4)}(a)=\alpha_4, \\ y(b)=\beta_0,\; y'(b)=\beta_1, y''(b)=\beta_2, y^{(3)}(b)=\beta_3 \end{cases}, \tag{1.1a}$$

where α_i,β_i (i = 0,…,3) and α_4 are finite real constants and $q_i(x)$ $(i=0,...,8)$ are all continuous functions on [a, b].

The second type, general nonlinear ninth-order initial value problems of the form:

$$y^{(9)}(x)=f(x,y,y',y''...,y^{(8)})\;,\; x\in\,[a,b], \tag{1.2}$$

with the following initial conditions:

$$y^{(i)}(a)=\alpha_i,\; i=0,1,...,8. \tag{1.2a}$$

Several spline collocation methods for solving high-order BVPs are presented by (Kasi et al, 6), (Lamnii et al ,7) and (Mahmoud, 9-11). Moreover, optimal homotopy asymptotic and homotopy perturbation methods for solving these problems are considered in [1,5, 12-14]. Hassan et al [3] and Hesaaraki and Jalilian [4] have been applied variational iteration methods of numerical solution for the proposed problem.

Hassan and Ertürk [2, 2009] is presented a numerical comparison between the differential transform method and Adomian decomposition method , as well, modified decomposition method is considered by Wazwaz in [15] for solving boundary value problems from the forms (1.2)-(1.2a).

Importance of Research and its Aim

It is well known that the analytical solutions of those higher-order BVPs are either very difficult or not existent. So, the numerical solutions of them are very important.

This work aims to present spline method with three collocation points for finding the numerical solutions for general linear ninth-order BVP (1.1)-(1.1a), and general nonlinear ninth-order IVPs (1.2)-(1.2a).

Methodology

The paper is organized as follows. In **section** 2, the ninth-order BVP (1.1)-(1.1a) is transformed into five initial value problems (IVPs). Polynomial splines with three collocation points are directly applied into ninth-order IVPs and then the spline numerical solution and its derivatives up to ninth order are computed. Moreover, polynomial splines are directly applied into nonlinear ninth-order IVPs for finding its spline numerical solution. **Section** 3, the existence and uniqueness of spline solution of the ninth-order BVP are proved. The convergence analysis and error estimation of the spline method are discussed. Finally, in **Section** 5 and 6, we conclude with numerical results, discussion and conclusion.

2- Spline Collocation Method

In this section, the ninth-order BVP is transformed into five IVPs. After that, spline functions are formulated to be applied directly into the five IVPs for finding the spline solution and its derivatives up to ninth-order of them.

2.1 Solution Scheme of ninth-order BVP

Consider the ninth-order BVP (1.1):

$$y^{(9)}(x) = -\sum_{i=0}^{8} q_i(x)\; y^{(i)}(x) + g(x),\;\; x \in\; [a, b], \tag{2.1}$$

subject to the following boundary conditions:

$$\begin{cases} y(a) = \alpha_0,\; y'(a) = \alpha_1, y''(a) = \alpha_2, y^{(3)}(a) = \alpha_3,\; y^{(4)}(a) = \alpha_4, \\ y(b) = \beta_0,\; y'(b) = \beta_1, y''(b) = \beta_2, y^{(3)}(b) = \beta_3 \end{cases}, \tag{2.1a}$$

Let $y(x)$ be the unique solution to the BVP (2.1)-(2.1a), then this solution is associated by a linear combination consists of five special IVPs. To find it, we assume that $U(x)$ is the unique solution to the following ninth-order IVP:

$$U^{(9)}(x) = -\sum_{i=0}^{8} q_i(x) U^{(i)}(x) + g(x), \quad a \le x \le b\;, \tag{2.2}$$

with the following initial conditions:

$$U^{(k)}(a) = \alpha_k,\;\; (k = 0,1...,4),\;\; U^{(k)}(a) = 0,\;\; (k = 5,...,8), \tag{2.2a}$$

In addition, suppose that $U_1(x)$, $U_2(x)$ $U_3(x)$ and $U_4(x)$ are the unique solutions to the following four homogeneous ninth-order IVPs, respectively, the first equation:

$$U_1^{(9)} = -\sum_{i=0}^{8} q_i(x) U_1^{(i)}(x), \quad a \le x \le b, \tag{2.3}$$

with the following initial conditions:

$$U_1^{(k)}(a) = 0,\;\; (k = 0,...,4),\;\; U_1^{(5)}(a) = 1,\, U_1^{(k)}(a) = 0\;\; (k = 6,7,8), \tag{2.3a}$$

The second equation

$$U_2^{(9)} = -\sum_{i=0}^{8} q_i(x) U_2^{(i)}(x), \quad a \le x \le b \tag{2.4}$$

with the following initial conditions:

$$U_2^{(k)}(a) = 0,\;\; (k = 0,...,5),\;\; U_2^{(6)}(a) = 1,\, U_2^{(k)}(a) = 0\;\; (k = 7,8), \tag{2.4a}$$

The third equation

$$U_3^{(9)} = -\sum_{i=0}^{8} q_i(x) U_3^{(i)}(x), \quad a \le x \le b, \tag{2.5}$$

with the following initial conditions:

$$U_3^{(k)}(a) = 0,\;\; (k = 0,...,6),\;\; U_3^{(7)}(a) = 1,\, U_3^{(8)}(a) = 0, \tag{2.5a}$$

The final fourth equation:

$$U_4^{(9)} = -\sum_{i=0}^{8} q_i(x) U_4^{(i)}(x), \quad a \le x \le b \tag{2.6}$$

with only the following initial conditions:

$$U_4^{(k)}(a) = 0,\;\; (k = 0,...,7),\;\; U_4^{(8)}(a) = 1, \tag{2.6a}$$

Then, for four real constants c_1, c_2, c_3, and c_4 there exists the linear combination:

$$y(x) = U(x) + \sum_{k=1}^{4} c_k U_k \qquad (2.7)$$

which it is a solution to the ninth-order BVP (2.1)-(2.1a), as seen by the following computations:

$$y^{(9)}(x) = U^{(9)}(x) + \sum_{k=1}^{4} c_k U_k^{(9)} =$$

$$= -\sum_{i=0}^{8} q_i(x) U^{(i)}(x) + g(x) + \sum_{k=1}^{4} c_k [-\sum_{i=0}^{8} q_i(x) U_k^{(i)}(x)]$$

$$= -\sum_{i=0}^{8} q_i(x)[U^{(i)}(x) + \sum_{k=1}^{4} c_k U_k^{(i)}(x)] + g(x) = -\sum_{i=0}^{8} q_i(x)\ y^{(i)}(x) + g(x),$$

where $y^{(i)}(x) = U^{(i)}(x) + \sum_{k=1}^{4} c_k U_k^{(i)}, i = 0,1,...,8$.

Now, it will be illustrated that the formulated solution $y(x)$ by equation (2.7) holds on the boundary values (2.1a), thus from conditions (2.1a) it yields out:

$$y^{(i)}(a) = U^{(i)}(a) + \sum_{k=1}^{4} c_k\, U_k^{(i)}(a) = \alpha_i + \sum_{k=1}^{4} c_k(0) = \alpha_i\,, \quad (i = 0,...,4)$$

The unknown constants c_1, c_2, c_3, and c_4 will be determined from the remainder of the end conditions by solving the system of equations:

$$y^{(i)}(b) = U^{(i)}(b) + \sum_{k=1}^{4} c_k\, U_k^{(i)}(b) \equiv \beta_i\,, \quad (i = 0,1,2,3), \qquad (2.8)$$

Now, since the proposed BVP (2.1)-(2.1a) has been reduced into five IVPs (2.2)-(2.6), spline techniques with three collocation points are applied for solving the ninth-order IVPs.

2.2 Formulation of the Spline Approximations.

Denote by $x_i = a + i\,h,\ i = 0,1,..., N$, the grid points of the uniform partition of $[a,b]$ into subintervals $\mathrm{I}_k = [x_k, x_{k+1}],\ k = 0,1,\ldots, N-1$, and $h = (b-a)/N$ is the constant stepsize. Let $S(x)$ be the spline approximation of the function $y(x)$ that can be represented on each I_k by:

$$S(x) = \sum_{i=0}^{8} \frac{(x-x_k)^i}{i!} S_k^{(i)} + \sum_{i=9}^{11} \frac{(x-x_k)^i}{i!} C_{k,i-8}\,, x \in [x_k, x_{k+1}],\ k = 0,...,N-1 \ , \qquad (2.9)$$

where $S^{(i)}(a) = S_0^{(i)}\ (i = 0,...,8)$.

The proposed method uses three collocation points:

$$x_{k+z_j} = x_k + h\ z_j\,,\ (j=1,2,3), \qquad (2.10)$$

with collocation parameters are given as

$$0 < z_1 < z_2 < z_3 = 1$$

To apply the spline approximation (2.9) and its derivatives up to ninth-order with respect to x, into ninth-order IVPs (2.2)-(2.6), to be satisfied with three collocation points (2.10), in each subinterval $\mathrm{I}_k = [x_k,\ x_{k+1}]$, k=0(1)N-1, then we have, respectively:

$$S_U^{(9)}(x_{k+z_j}) = -\sum_{i=0}^{8} g_i(x_{k+z_j}) S_U^{(i)}(x_{k+z_j}) + g(x_{k+z_j}) \ ,\quad j = 1,2,3,\quad k = 0(1)N-1, \qquad (2.11)$$

with the following initial conditions:

$$S_U^{(i)}(a)=\alpha_i \ (i=0,...,4), \quad S_U^{(i)}(a)=0 \ (i=,5,...,8), \tag{2.11a}$$

$$S_{U_1}^{(9)}(x_{k+z_j})=-\sum_{i=0}^{8} q_i(x_{k+z_j})S_{U_1}^{(i)}(x_{k+z_j}) \ , \quad j=1(1)3, \quad k=0(1)N-1, \tag{2.12}$$

with the following initial conditions:

$$S_{U_1}^{(i)}(a)=0 \ (i=0,...,4), \ S_{U_1}^{(5)}(a)=1, \ S_{U_1}^{(i)}(a)=0 \ \ (i=6,7,8), \tag{2.12a}$$

$$S_{U_2}^{(9)}(x_{k+z_j})=-\sum_{i=0}^{8} q_i(x_{k+z_j})S_{U_2}^{(i)}(x_{k+z_j}) \ , \quad j=1(1)3, \quad k=0(1)N-1, \tag{2.13}$$

with the following initial conditions:

$$S_{U_2}^{(i)}(a)=0 \ (i=0,...,5), \ S_{U_2}^{(6)}(a)=1, \ S_{U_2}^{(i)}(a)=0 \ \ (i=7,8), \tag{2.13a}$$

$$S_{U_3}^{(9)}(x_{k+z_j})=-\sum_{i=0}^{8} q_i(x_{k+z_j})S_{U_3}^{(i)}(x_{k+z_j}) \ , \quad j=1(1)3, \quad k=0(1)N-1, \tag{2.14}$$

with the following initial conditions:

$$S_{U_3}^{(i)}(a)=0 \ (i=0,...,6), \ S_{U_3}^{(7)}(a)=1, \ S_{U_3}^{(8)}(a)=0, \tag{2.14a}$$

Finally, applying to five IVP:

$$S_{U_4}^{(9)}(x_{k+z_j})=-\sum_{i=0}^{8} q_i(x_{k+z_j})S_{U_4}^{(i)}(x_{k+z_j}) \ , \quad j=1(1)3, \quad k=0(1)N-1, \tag{2.15}$$

with only the following initial conditions:

$$S_{U_4}^{(i)}(a)=0 \ (i=0,...,7), \ S_{U_4}^{(8)}(a)=1 \ . \tag{2.15a}$$

Now, by substituting spline solutions to the system of linear equations (2.8), the coefficients c_1, c_2, c_3, and c_4 that associate with boundary conditions , will be known as follow:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} S_{U_1}(b) & S_{U_2}(b) & S_{U_3}(b) & S_{U_4}(b) \\ S'_{U_1}(b) & S'_{U_2}(b) & S'_{U_3}(b) & S'_{U_4}(b) \\ S''_{U_1}(b) & S''_{U_2}(b) & S''_{U_3}(b) & S''_{U_4}(b) \\ S^{(3)}_{U_1}(b) & S^{(3)}_{U_2}(b) & S^{(3)}_{U_3}(b) & S^{(3)}_{U_4}(b) \end{bmatrix}^{-1} \begin{bmatrix} \beta_0 - S_U(b) \\ \beta_1 - S'_U(b) \\ \beta_2 - S''_U(b) \\ \beta_3 - S^{(3)}_U(b) \end{bmatrix}$$

Thus, the spline solutions $S^{(i)}(x), i=0,...,9$ of the BVP (2.1)-(2.1a), will be known by:

$$S^{(i)}(x_k)=S_U^{(i)}(x_k)+\sum_{j=1}^{4} c_j S_{U_j}^{(i)}(x_k), \quad i=0,1,...,9 \, . \tag{2.16}$$

Moreover, applying collocation points $x_{k+z_j}=x_k+h\,z_j$, (j=1,2,3) to (2.9), we obtain

$$S(x_{k+z_j})=\sum_{i=0}^{8}\frac{(h\,z_j)^i}{i!}S_k^{(i)}+\sum_{i=9}^{11}\frac{(h\,z_j)^i}{i!}C_{k,i-8}\,, \quad j=1,..,3, \quad k=0,...,N-1. \tag{2.17}$$

where $\quad z_j = j/3$, $\quad x_{k+z_j} \in]x_k, x_{k+1}]$, ($j$=1,2,3).

The first three coefficients $C_{k,1}, C_{k,2}, C_{k,3}$ are computed from the linear systems (2.11)-(2.15) by using the initial value conditions if k=0, or from the previous steps if k>1.

2.3 Spline Solution of nonlinear ninth-order IVP

The numerical solutions of nonlinear ninth-order IVPs by proposed spline method are more easy than BVPs, so the spline approximation (2.9) and its derivatives $S^{(i)}(x), i=0,...,8$, will be applied directly without reducing the problem to system of first-order differential equations. Now, spline collocation method is applied into (1.2)-(1.2a), to be satisfied with collocation points (2.10), in each subinterval $\mathrm{I}_k=[x_k, x_{k+1}]$, as follow:

$$S^{(9)}(x_{k+z_j})=f[x_{k+z_j}, S(x_{k+z_j}), S'(x_{k+z_j}),...,S^{(8)}(x_{k+z_j})], \quad x_{k+z_j}\in I_k, \qquad k=0(1)N\text{-}1$$

with the following initial conditions:

$$S_0^{(i)}(a)=\alpha_i,\ i=0,1,...,8.$$

2.4 Stability of Spline Collocation Method

Consider the following ninth-order IVP:

$$\begin{cases} y^{(9)}(x)=F(x,y(x),...,y^{(8)}(x)), & x\in[a,b] \\ y^{(d)}(a)=a_d, & d=0(1)8. \end{cases} \qquad (2.18)$$

Suppose that $F:[a,\ b]\times C[a,\ b]\times\cdots\times C^7[a,\ b]\to R$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last argument:

$$|F(x,y_0,...,y_8)-F(x,\ddot{y}_0,...,\ddot{y}_8)|\le L\sum_{i=0}^{8}|y_i-\ddot{y}_i|, \forall\ (x,y_0,...,y_8),(x,\ddot{y}_0,...,\ddot{y}_8)\in[a,\ b]\times R^9$$

where the constant L is called a Lipschitz constant for F.

These conditions ensure the existence of a unique solution $y(x)$ of problem (2.18).

By applying the Spline approximations (2.9) and its derivatives into the problem (2.18), to be satisfied with three collocation points (2.10), we obtain the linear system:

$$C_{k,1}+(hz_j)C_{k,2}+\frac{(hz_j)^2}{2}C_{k,3}=F(x_{k+z_j},\ S(x_{k+z_j}),...,S^{(8)}(x_{k+z_j})),\ j=1,..,3,$$
$$k=0,...,N-1, \qquad (2.19)$$

$$S^{(d)}(a)=a_d,\ \ d=0(1)8. \qquad (2.20)$$

We rewrite (2.19) in the matrices form:

$$A\overline{C}_k=\hat{F}_k \qquad (2.21)$$

where

$$A=\begin{bmatrix} 1 & hz_1 & \frac{h^2z_1^2}{2!} \\ 1 & hz_2 & \frac{h^2z_2^2}{2!} \\ 1 & h & \frac{h^2}{2!} \end{bmatrix},\quad \overline{C}_k=\begin{bmatrix} C_{k,1} \\ C_{k,2} \\ C_{k,3} \end{bmatrix},\quad \hat{F}_k=\begin{bmatrix} F_{k+z_1} \\ F_{k+z_2} \\ F_{k+1} \end{bmatrix}, \ ,$$

$$F_{k+Z_j}=F(x_{k+Z_j},\ S(x_{k+Z_j}),\ldots,S^{(8)}(x_{k+Z_j})), \quad j=1,2,3.$$

<u>Definition 1</u> [**9**]: The spline collocation method (2.21) is called stable if eigenvalues of the matrix A satisfy

$$|\mu_j|<1,\ j=1,...,m$$

where A is the matrix in the linear system .

Corollary 1. The spline collocation method applied to (2.18) is stable if eigenvalues of the matrix A satisfy

$$|\mu_1|,\ |\mu_2|,|\mu_3|<1, \tag{2.22}$$

for $z_j = j/3$ (j=1,2,3), where A is the matrix in relation (2.21) .

Proof. The spline method applied to ninth-order IVP is stable if conditions $|\mu_j|<1,\ j=1,2,3$ are satisfied. To do this, we find that the matrix A has the three different eigenvalues $\mu_1 = 0.545904, \mu_2 = 0.160679, \mu_3 = 0.0156385$, for $z_1 = 1/3$, $z_2 = 2/3,\ z_3 = 1$.

3 Convergence Analysis and Error Estimation

Here, we find to introduce two following definitions.

Definition 2 [9]: A spline collocation method is said to be consistent of order p if $\max\limits_{1\le k\le N} \|\bar{\tau}_k\| = O(h^p)$, where $\bar{\tau}_k$ is local discretization error at x_i.

Definition 3 [9]: The spline collocation method is said to be convergence if $\lim\limits_{h\to 0} \max\limits_{1\le k\le N} |y(x_k) - S_k| = 0$, where $y(x_k)$ is the exact solution and S_k is the spline solution of spline collocation method at x_i.

We assume that $y(x) \in C^{12}[a, b]$, the unique solution of the linear ninth-order BVP and S(x) be a spline approximation solution to $y(x)$, also $\bar{\tau}_k$ is a 3-dimensional column vector. Here, the vector $\bar{\tau}_k$ is the local truncation error. Applying the Spline solution $S(x)$ on three collocation points $x_{k+z_j} = x_k + z_j h$, (j=1,2,3), putting $y(x_{k+z_j}) = y(x_k + hz_j)$, $S_k^{(m)} = S^{(m)}(x_k)$ and $y_k^{(m)} = y^{(m)}(x_k)$, (m=0,…,8), k=0,…,N-1, for $z_j = j/3$ (j=1,2,3), we obtain the local truncation error formula:

$$\bar{\tau}_k = \mathrm{M}\,\bar{C}_k + \bar{\Psi}_k, \tag{3.1}$$

where

$$\bar{\Psi}_k = \begin{bmatrix} \sum\limits_{i=0}^{8} \frac{(z_1 h)^i}{i!} S_k^{(i)} - y(x_k + z_1 h) \\ \sum\limits_{i=0}^{8} \frac{(z_2 h)^i}{i!} S_k^{(i)} - y(x_k + z_2 h) \\ \sum\limits_{i=0}^{8} \frac{h^i}{i!} S_k^{(i)} - y(x_k + h) \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} \frac{(z_1 h)^9}{9!} & \frac{(z_1 h)^{10}}{10!} & \frac{(z_1 h)^{11}}{11!} \\ \frac{(z_2 h)^9}{9!} & \frac{(z_2 h)^{10}}{10!} & \frac{(z_2 h)^{11}}{11!} \\ \frac{h^9}{9!} & \frac{h^{10}}{10!} & \frac{h^{11}}{11!} \end{bmatrix},$$

$$\bar{C}_k = \begin{bmatrix} C_{k,1} \\ C_{k,2} \\ C_{k,3} \end{bmatrix}$$

On the other hand, from the system (2.21), we get

$$\bar{C}_k = A^{-1} \hat{F}_k \tag{3.2}$$

where A^{-1} is the matrix (2.20), and $\hat{F}_k = [y^{(9)}(x_{k+z_1}) \;\; , y^{(9)}(x_{k+z_2}) \;\; , y^{(9)}(x_{k+1})]^T$.

Using Taylor's expansions for the functions $y^{(m)}(x), m = 0,...,9$ about x_k, in the relation (3.2) and substituting into (3.1), we get the local truncation error at the *k*th step as follows:

$$\bar{\tau}_k = \mathrm{M}(A^{-1}\hat{F}_k) + \overline{\Psi}_k = \begin{bmatrix} \frac{367}{177147*12!} h^{12} y^{(12)}(x_k) \\ \frac{2}{1076168025} h^{12} y^{(12)}(x_k) \\ \frac{29}{12*11!} h^{12} y^{(12)}(x_k) \end{bmatrix} \quad ,k=0,1,..,N \qquad (3.3)$$

where

$$y(x) = \sum_{i=0}^{12} \frac{(x-x_k)^i}{i!} y^{(i)}(x_k) + O(h^{13}),\; x \in [x_k, x_{k+1}] \; .$$

Note from the relation (3.3) that the local truncation error of the presented Spline collocation method is $\| \bar{\tau}_k \|_\infty = \frac{29}{12*11!} y^{(12)}(x_k) h^{12} \equiv O(h^{12})$ and thus the global error after *N* steps will be $\max_{1\le k\le N} \| \bar{\tau}_k \| = N.O(h^{12}) = \frac{b-a}{h}.O(h^{12}) \equiv O(h^{11})$, we deduce, according to Definition 2, that the method is thus consistent and is of order at least eleven for $z_j = j/3$ (*j*=1,2,3).

Consequently, we have obtained the following: let $y \in C^{11}[a, b]$ be Lipschitz continuous, then the spline approximation $S(x)$ converges to the solution *y*(*x*) of the ninth-order BVP as $h \to 0$ for $z_j = j/3$ (*j*=1,2,3) and

$$\lim_{h\to 0} S^{(m)}(e) = y^{(m)}(e),\; m = 0,...,8, \;\; e = a,\, b \; . \qquad (3.4)$$

Furthermore, the method is convergent according to Definition 3, for the reason that

$$\lim_{h\to\infty} \max_{0\le k\le N} \| \bar{\tau}_k \|_\infty = \lim_{h\to\infty} O(h^9) = 0 \Rightarrow \lim_{h\to 0} \max_{1\le k\le N} | y(x_k) - S_k | = 0$$

4. Numerical Results and Discussions

The experiments below are designed to test the efficiency of the spline method when applied with three collocation points to linear and nonlinear ninth-order BVPs with uniform grids. These problems have exact solutions, thus we compute their actual errors. In calculations, the notations $\delta^{(k)} = \max \| y^{(k)}(x) - S^{(k)}(x) \|$ are used to denote maximum absolute errors, where , *k*=0,1,..,8 indicate orders of derivatives. Numerical results of examples are obtained from computer programs designed by *TPW* 1.5 in double precision, and figures are plotted by *Mathematica* 9.

Problem 1 . Consider the ninth-order linear BVP (cf. [1, 5, 14,15]):

$$y^{(9)}(x) - y(x) = -9\,\exp(x), \quad 0 \le x \le 1, \qquad (4.2)$$

with boundary conditions:

$$\begin{cases} y(0) = 1,\; y'(0) = 0,\; y''(0) = -1,\; y^{(3)}(0) = -2, y^{(4)}(0) = -3 \\ y(1) = 0,\; y'(1) = -e\;, y''(1) = -2e, y^{(3)}(1) = -3e, \end{cases} , \qquad (4.2a)$$

Its exact solution is $y(x)=(1-x)\exp(x)$. In **Table1**, the maximum absolute errors in the spline solution and its derivatives up to ninth order are calculated by proposed spline method for N=10. **Table 2** and **Table 3** illustrate comparisons between the absolute errors obtained by presented spline method and other methods in [1, 5, 14,15]. The spline solutions $S(x)$, $S^{(4)}(x)$, $S^{(8)}(x)$ and the exact solutions as well as absolute errors in $S^{(8)}(x)$ are illustrated in **Figs.1-4,** respectively.

Table 1: Maximum absolute errors of Problem 1, for *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$
1.4E-15	6.2E-15	5.4E-14	7.6E-13	4.0E-12
$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$	$\delta^{(8)}$	$\delta^{(9)}$
6.7E-11	7.7E-10	5.3E-9	2.1E-8	1.4E-15

Table 2: Comparison of maximum absolute errors of the present method with other methods, for Problem1, in $y_i^{(m)}, m=0,1,...,9$.

$y_i^{(m)}$	Least square method [7]	Variational itera. Method [3]	Presented Spline Method
y_i	0.1231 E-19	9.08 E-12	3.0358 E−18
y_i'	0.5848E-18	-------	5. 3535 E−17
$y_i^{(2)}$	0.4180E-16	9.02E-11	1.3661 E−17
$y_i^{(3)}$	0.3599E-14	-------	1. 0716 E−15
$y_i^{(4)}$	0.5722E-12	2.57 E-09	5. 3535 E−14
$y_i^{(5)}$	0.2058E-09	-------	2.7252 E−13
$y_i^{(6)}$	0.2765E-07	1.71 E-06	3.5138 E−12
$y_i^{(7)}$	0.1160E-05	-------	4.1875 E−11
$y_i^{(8)}$	0.1670E-04	1.83 E-04	4.9239 E−09
$y_i^{(9)}$	0.6209E-01	-------	4.2268 E-18

Table 3: The absolute errors of Problem 1.

x_i	Homotopy Method[5]	Homotopy Perturbation Method[14]	Homotopy Asymptotic Method[1]	Modified Decomposition Method [15]	Presented Spline method h=1/10
0.1	3.6E –0 9	2.0E-10	$1.03E-16$	2.0E-10	6.2342E-18
0.2	3.4E –09	2.0E-10	$1.33E-16$	2.0E-10	8.8146E-17
0.3	4.6E – 09	2.0E-10	$2.12E-16$	2.0E-10	3.8836E-16
0.4	1.4E – 09	2.0E-10	$1.30E-14$	2.0E-10	9.0222E-16
0.5	4.5E – 09	2.0E-10	$2.44E-13$	2.0E-10	1.3451E-15
0.6	6.0E – 06	6.0E-10	$2.64E-12$	6.0E-10	1.3618E-15
0.7	3.1E – 09	1.0E-09	$1.97E-11$	1.0E-09	8.9154E-16
0.8	2.4E – 09	2.0E-09	$1.13E-10$	2.0E-09	2.9626E-16
0.9	4.5E –09	3.4E-09	$5.26E-10$	3.4E-09	1.2468E-17
1.0	0.000000	0.000000	$2.09E-09$	0.000000	5.3595E-19

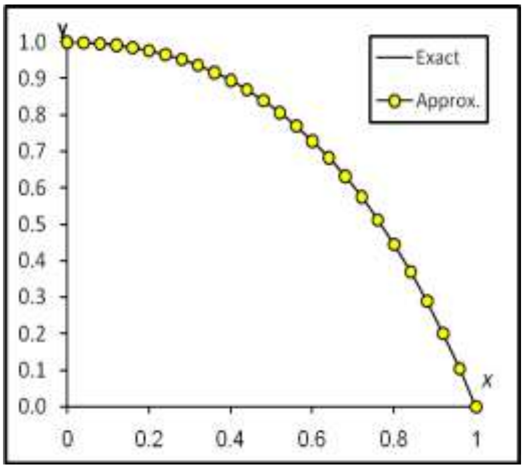

*Fig.*1: *The spline solution S*(x) *and the exact solution y*(x), *for Problem1*, h=1/25.

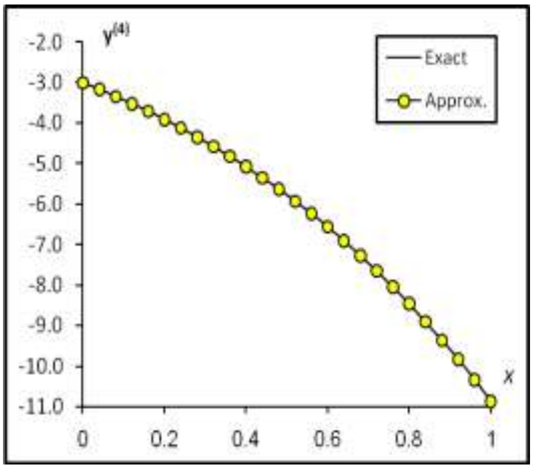

*Fig.*2: *The spline solution* $S^{(4)}$ (x) *and the exact solution* $y^{(4)}(x)$, *for Problem 1*, for N=25.

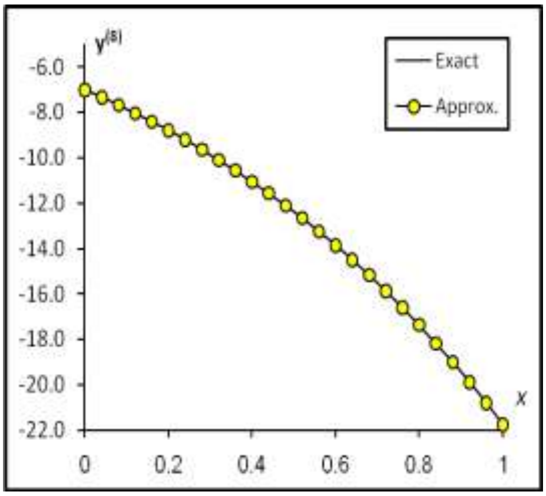

*Fig.*3: *The spline solution* $S^{(8)}$ (x) *and the exact solution* $y^{(8)}(x)$, *for Problem 1*, for N=25.

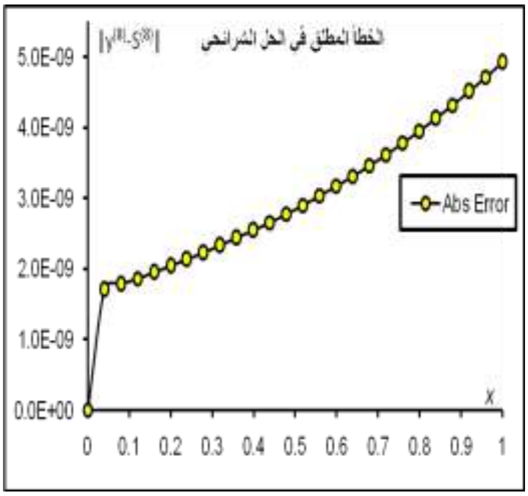

*Fig.*4: *The absolute error in spline solution* $S^{(8)}(x)$, for N=25.

Problem 2. We consider the following nonlinear BVP (cf. [1]):

$$\begin{cases} y^{(9)} = \exp(-x)\, y^{(2)}(x), \qquad 0 \le x \le 1, \\ \quad y(0) = y'(0) = y''(0) = \cdots = y^{(8)}(0) = 1. \end{cases}$$

The exact solution is $y(x) = \exp(x)$. **Table 4** appears comparisons of the numerical solution and absolute errors by presented spline method with other by the optimal homotopy asymptotic method [**1**]. In **Table5**, the maximum absolute errors in the spline solution and its derivatives up to ninth order are calculated by proposed spline method for N=10.

Table 4: The numerical solution and absolute errors of nonlinear problem 2, for *h*=10.

x_i	*Homotopy method* [1, 2012]			Presented spline methods	
	Exact solution	Homo. Sol.	Abs Error	Spline Sol.	Abs Error
0.1	1.1051709180756	1.105170915	$1.39E-16$	1.1051709180756	1.084E-19
0.2	1.2214027581602	1.221402732	$5.32E-16$	1.2214027581602	5.141E-16
0.3	1.3498588075760	1.349858729	$1.48E-15$	1.3498588075760	4.790E-15
0.4	1.4918246976413	1.491824562	$3.36E-15$	1.4918246976413	1.995E-14
0.5	1.6487212707001	1.648721109	$2.28E-14$	1.6487212707001	5.783E-14
0.6	1.8221188003905	1.822118662	$2.21E-13$	1.8221188003904	1.362E-13
0.7	2.0137527074705	2.013752625	$1.64E-12$	2.0137527074702	2.800E-13
0.8	2.2255409284925	2.225540900	$9.36E-12$	2.2255409284920	5.240E-13
0.9	2.4596031111569	2.459603108	$4.36E-11$	2.4596031111561	9.149E-13
1.0	2.7182818284590	2.718281828	$1.73E-10$	2.7182818284576	1.516E-12

Table 5: Maximum abs errors of Problem 2, for *h*=0.1.

$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$
1.4E-12	6.6E-12	2.3E-11	6.6E-11	3.2E-11
$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$	$\delta^{(8)}$	$\delta^{(9)}$
1.6E-10	8.8E-10	2.8E-9	5.3E-9	2.8E-12

Problem 3. Consider the following general ninth-order BVP:

$$y^{(9)}(x)+\sum_{i=0}^{8} y^{(i)}(x)=(x-4)\cos(x)+(x+5)\sin(x),$$

$$y(-\pi)=0,\ y'(-\pi)=\pi, y''(-\pi)=-2, y^{(3)}(-\pi)=-\pi, y^{(4)}(-\pi)=4$$

$$y(\pi)=0\ ,\quad y'(\pi)=-\pi,\quad y''(\pi)=-2,\ y^{(3)}(\pi)=\pi.$$

The exact solution is $y(x)=x\sin(x)$. In **Table6,** the absolute errors in the spline solution and its derivatives up to ninth order are calculated by proposed spline method. The spline solutions $S(x)$,$S'''(x)$, $S^{(6)}(x)$ as well as the solutions $y(x)$, ,$y'''(x)$, $y^{(6)}(x)$ are illustrated in **Figs.5-14,** respectively.

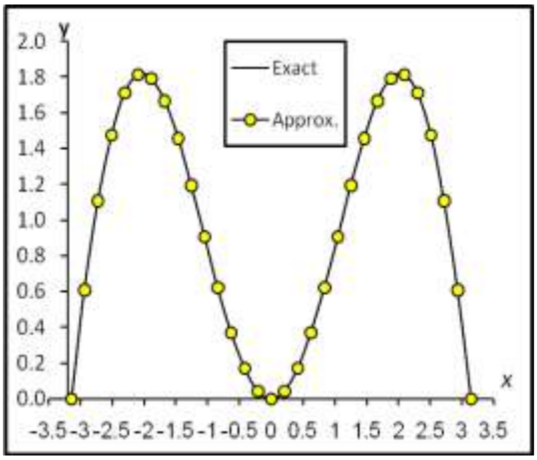

***Fig.*5**: *The spline solution S*(*x*) *and the exact solution y*(*x*), *for Problem4, h*=Pi/15.

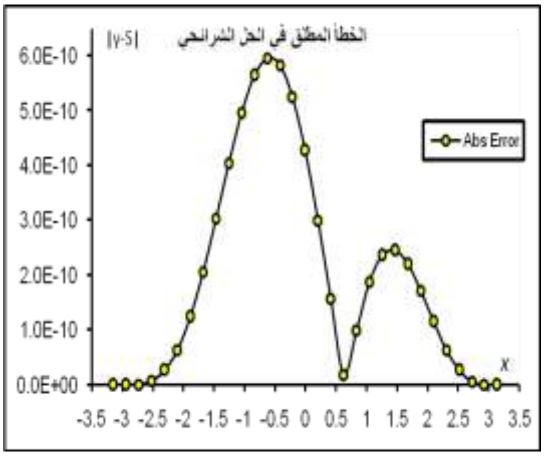

***Fig.*6**: *The absolute error in spline solution S*(*x*) , for *N*=30.

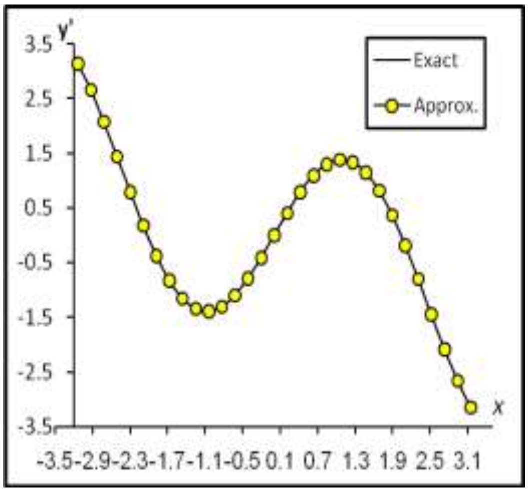

*Fig.*7: *The spline solution* S'(*x*) *and the exact solution* y'(*x*), *for Problem 3, h*=Pi/15.

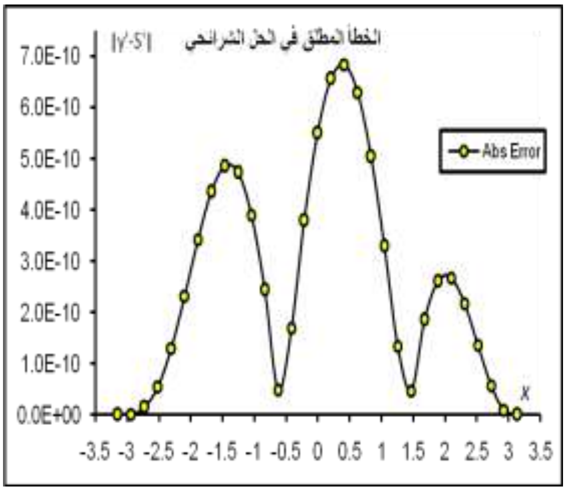

*Fig.*8: *The absolute error in spline solution* S'(*x*) , for *N*=30.

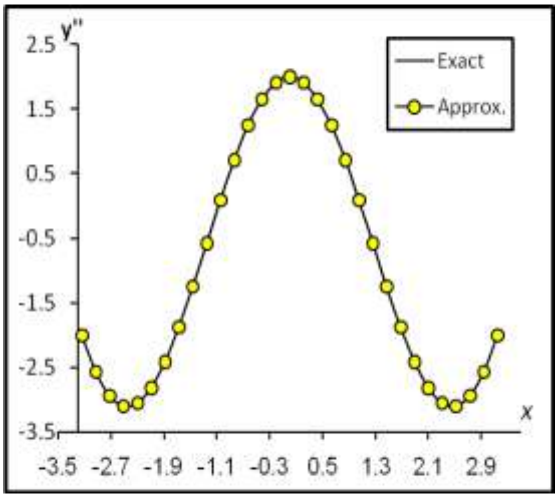

*Fig.*9: *The spline solution* S''(*x*) *and the exact solution* y'(*x*), *for Problem 3, h*=Pi/15.

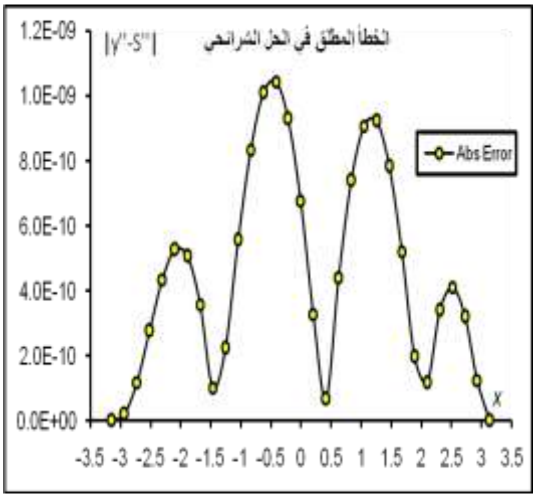

*Fig.*10: *The absolute error in spline solution* S''(*x*) , for *N*=30.

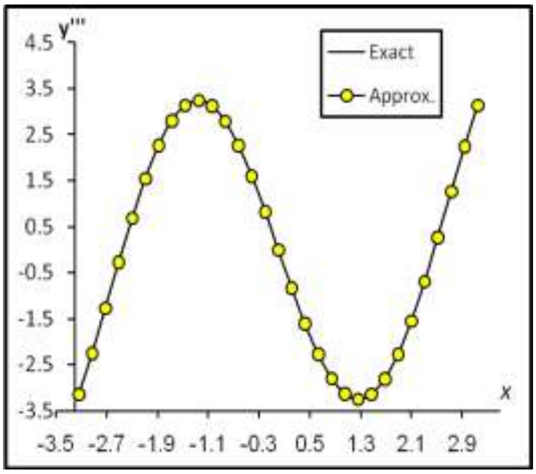

***Fig.*11**: *The spline solution* S'''(x) *and the exact solution* y'''(x), *for Problem 3, h*=Pi/15.

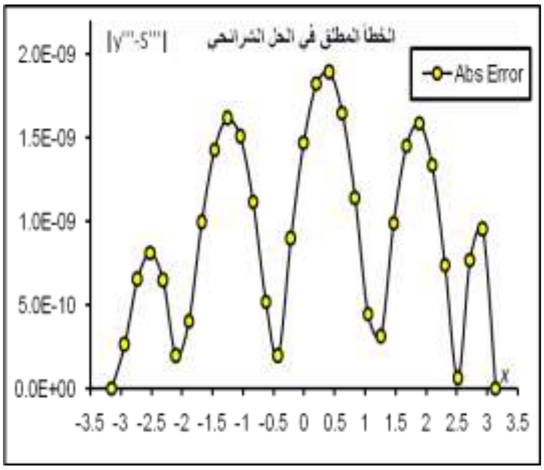

***Fig.*12**: *The absolute error in spline solution* S'''(x) , for N=30.

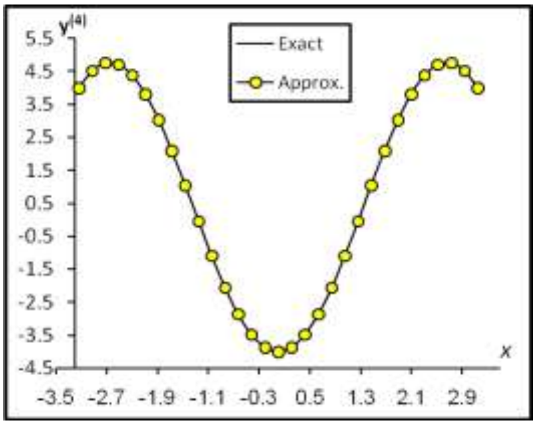

***Fig.*13**: *The spline solution* $S^{(4)}$ (x) *and the exact solution* $y^{(4)}$(x), *for Problem 3, h*=Pi/15.

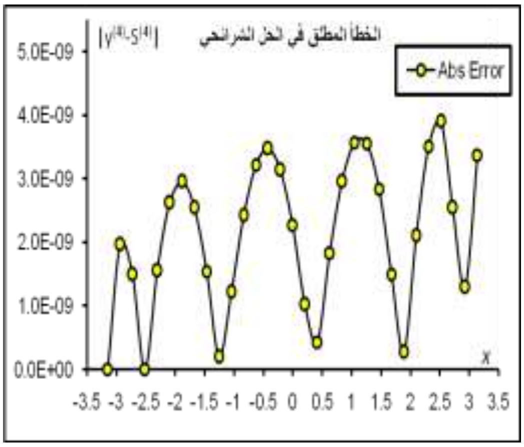

***Fig.*14**: *The absolute error in spline solution* $S^{(4)}$(x) , for N=30.

Table 6: The absolute errors of Problem 3, by the Presented Spline Method, for *N*=32.

x_i	$\delta^{(0)}$	$\delta^{(1)}$	$\delta^{(2)}$	$\delta^{(3)}$	$\delta^{(4)}$
$-\frac{3\pi}{8}$	1.8201E-14	8.4791E-14	1.905E-13	5.617E-12	4.804E-11
$-\frac{\pi}{4}$	7.939E-14	3.674E-13	9.968E-13	1.005E-13	3.398E-11
$-\frac{\pi}{8}$	1.933E-13	1.861E-13	1.985E-12	1.091E-12	6.782E-11
0	1.741E-13	7.977E-13	7.416E-13	1.9891E-12	3.917E-11
$\frac{\pi}{8}$	1.969E-13	5.974E-13	8.474E-13	7.852E-12	3.958E-11

$\frac{\pi}{4}$	5.168E-14	6.596E-13	9.931E-14	5.458E-12	4.261E-11
$\frac{3\pi}{8}$	3.894E-14	9.998E-14	4.735E-13	2.222E-12	5.008E-11
$\frac{\pi}{2}$	2.895E-15	5.221E-15	5.223E-15	6.2881E-15	5.572E-11
x_i	$\delta^{(5)}$	$\delta^{(6)}$	$\delta^{(7)}$	$\delta^{(8)}$	$\delta^{(9)}$
$-\frac{3\pi}{8}$	7.743E-11	8.769E-11	8.829E-11	2.833E-10	5.836E-11
$-\frac{\pi}{4}$	9.559E-11	8.700E-11	8.914E-11	1.176E-10	5.964E-11
$-\frac{\pi}{8}$	8.262E-11	3.669E-11	9.507E-11	6.677E-11	6.663E-12
0	7.393E-11	5.873E-11	9.895E-11	1.023E-10	4.542E-11
$\frac{\pi}{8}$	4.377E-11	6.783E-11	7.201E-11	7.869E-11	6.522E-11
$\frac{\pi}{4}$	3.959E-11	7.878E-11	8.789E-11	8.794E-11	7.794E-11
$\frac{3\pi}{8}$	3.295E-11	8.586E-11	8.975E-11	1.011E-10	8.647E-11
$\frac{\pi}{2}$	8.348E-11	8.311E-11	8.862E-11	4.220E-10	9.992E-11

Rate of Convergence: Here, the order of convergence is computed when the Spline collocation method applied to following nonlinear test problem:

$$y^{(9)} = \frac{2835}{4}(x+y+1)^9, \qquad 0 \le x \le 1,$$

$$y(0)=0, y'(0)=-0.5, y''(0)=0.5, y'''(0)=0.75, y^{(4)}(0)=1.5,$$

$$y^{(5)}(0)=15/4, y^{(6)}(0)=45/4, y^{(7)}(0)=315/8, y^{(8)}(0)=318/2.$$

The exact solution is $y(x)=\frac{2}{(2-x)}-x-1$.

The nodal difference error ε_k^N, is defined by:

$$\varepsilon_k^N = |S_k^N - S_{2k}^{2N}|, \quad k=1,\ldots,N$$

where S_k^N is the spline solution at x_k by the present spline method. The experimental nodal rate of convergence is given by $Rate = Log_2(\varepsilon_k^N / \varepsilon_{2k}^{2N})$.

Table 7 shows spline solutions of test problem in the interval [0,1], for ***N***=10, 20, 40 by presented spline method. The order of convergence for the proposed spline method is computed in **Table 8**.

Table 7: ***The local errors for test problem by presented spline method for N*=10**

k	S_k^N	S_{2k}^{2N}	S_{4k}^{4N}
1	-0.047368421052632	-0.047368421052632	-0.047368421052632
2	-0.088888888888891	-0.088888888888889	-0.088888888888889
3	-0.123529411764720	-0.123529411764706	-0.123529411764706

4	-0.150000000000062	-0.150000000000001	-0.150000000000000
5	-0.166666666666884	-0.166666666666671	-0.166666666666667
6	-0.171428571429240	-0.171428571428586	-0.171428571428573
7	-0.161538461540392	-0.161538461538509	-0.161538461538468
8	-0.133333333338732	-0.133333333333486	-0.133333333333356
9	-0.081818181833051	-0.081818181818661	-0.081818181818258
10	-0.000000000040644	-0.000000000001459	-0.000000000000244

Table 8: The rate of convergence for presented spline method , with N=10.

k	$\varepsilon_k^N = \lvert S_k^N - S_{2k}^{2N} \rvert$	$\varepsilon_{2k}^{2N} = \lvert S_{2k}^{2N} - S_{4k}^{4N} \rvert$	Rate of Convergence
1	1.04083408 E-16	1.04083408 E-18	6.64386
2	1.99840144 E-15	2.35722573 E-17	6.40561
3	1.40026878 E-14	1.52655665 E-16	6.51928
4	6.10067552 E-14	4.99600366 E-16	6.93205
5	2.13024042 E-13	1.99840144 E-15	6.73603
6	6.54004628 E-13	6.49480469 E-15	6.65387
7	1.88299376 E-12	2.05113703 E-14	6.52046
8	5.24602583 E-12	6.4989680 E-14	6.33487
9	1.43900030 E-11	2.01498543 E-13	6.15815
10	3.91855476 E-11	6.07575463 E-13	6.01127

Notice: the results in the Table 8 show that the rate of convergence for presented spline method bigger than six.

5. Conclusion

Collocation spline method is successfully applied with three collocation points for the numerical solution of linear and nonlinear ninth-order boundary value problems. The presented spline method is experienced on three test problems. Comparisons of the results obtained by the present spline method with obtained by Homotopy Asymptotic Method[1], Variational iteration Method [3], Homotopy Method[5], Least square method [7], Homotopy Perturbation Method[14] and Modified Decomposition Method [15] appear that the present method is very effective and is better than other methods

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