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Discrete Stochastic Integration

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☐ ABSTRACT ☐

We present in this article a game of chance (Saint Petersburg Paradox) and generalize it on a probability space as an example of a previsible (predictable) process, from which we get a discrete stochastic integration (DSI). Then we define a martingale X and present it as a good integrator of a discrete stochastic integration $\int C.dX$, which is called the martingale transform of X by C such that C is a previsible process.

After that we present the most important properties of the DSI, which include that the DSI is also a martingale , the theorem of stability for it, the definition of the covariation of two given martingales and the proof that the DSI is centered with a specific given variance.

Finally, we define Doob-decomposition and the quadratic variation and present Itô-formula as a certain sort of it.

Keywords: Martingale, previsible (predictable) process, discrete stochastic integration, local bounded, stability, covariation, quadratic variation, centered, Doob-decomposition, Itô-formula.

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التكامل العشوائي المنقطع

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□ ملخّص □

نقدم في هذه المقالة لعبة حظ (لعبة بيترسبورغ) و نعممها على فضاء احتمالي كمثال عن عملية قابلة للتوقع و التي من خلالها نحصل على تكامل عشوائي منقطع بعد ذلك نعرف المارتينجال ونقدمه كعنصر تفاضلي جيد للتكامل العشوائي المنقطع الذي يدعى تحويل المارتينجال بواسطة عملية قابلة للتوقع.

بعد ذلك نقدم أهم خصائص التكامل العشوائي المنقطع التي تتضمن بأن التكامل العشوائي المنقطع هو من جديد مارتينجال كما تشرح نظرية الاستقرار له و تعرف تباين مارتينجالين معطيين و تبين أن التكامل العشوائي متمركز بتباين محدد معطى.

أخيراً نعرف تقسيم دوب والتباين التربيعي ونقدم صيغة العالم إتو كنوع محدد من تقسيم دوب .

الكلمات المفتاحية: مارتينجال، عملية قابلة للتوقع، تكامل عشوائي منقطع، محدودة بشكل محلي، استقرار، اختلاف، تباين تربيعي، متمركز، تقسيم دوب، صيغة إتو.

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Introduction:

In this article we present some ideas about the theory of modern stochastic integration. The novelty is that we will define a martingale as a " good integrator „ of the discrete stochastic integration, which we will talk about in detail.

Firstly we present two examples about a previsible process, like games of chance (Petersburg game, Saint Petersburg Paradox), which leads us to a discrete stochastic integration (DSI).

Secondly we explain the most important properties of the DSI.

Thirdly we define Doob-decomposition and the quadratic variation and then present the discrete Itô-formula as a certain sort of it.

The importance of this research and its aims:

The expression $C.X$, the martingale transform of X by C, is the discrete analouge of the stochastic integral $\int C.dX$.

The way of research and its materials:

We used some scientific articles and some modern books about probability theory.

1. Examples about a previsible process:

1.1. The first Example about a previsible process:

We consider a game of chance in a casino, in which in every round, the stake which the player has chosen either is doubled repaid or get lost (is lost).

This is like the case of Roulette, where the player for example, can choose "red „ and if he gets a red number, he will win his stake back doubled, otherwise he will lose it.

There are 37 fields, of which 18 are red, 18 black and one is green (the zero).

The chance to win should be then $p = \frac{18}{37} < \frac{1}{2}$.

This game of chance is accomplished infinitely often independent behind each other.

Definition 1: We can generalize this game on a probability space $(\Omega, \mathcal{A}, p)$, such that $\Omega = \{-1,1\}^{\mathbb{N}}, \mathcal{A} = \left(2^{\{-1,1\}}\right)^{\otimes\mathbb{N}}$ the power set of Ω and the product measure is

$\mathbb{P} = ((1-p).\delta_{-1} + p.\delta_1)^{\otimes\mathbb{N}}$ with δ_a: $\underbrace{\beta(\mathbb{R})}_{Borel\ \sigma-Algebra} \rightarrow [0,1]; a\epsilon\mathbb{R}$ such that:

$$\delta_a(A) = \begin{cases} 1\ ; a \in A \\ 0\ ; a \notin A \end{cases}$$

We denote with: $D_n: \Omega \rightarrow \{-1,1\} : \omega \rightarrow D_n(\omega) = \omega_n$ to the result of the $n-th$ round for every $n \in \mathbb{N}$.(Projection on the $n-th$ component) then holds: $p(D_n = 1) = p$ and $p(D_n = -1) = 1 - p$. If the player does the random stake C_i in the $i-th$ round, the sum of the gainings after the $n-th$ round will be $S_n = \sum_{i=1}^{n} C_i.D_i$.

We suppose now, that the player follows the following strategy:

1. The first stake in the first round is $C_1 = 1$.
2. If the player wins, he will pay no more in the following games, that means: $C_n = 0$ for every $n \geq 2$, in the case of $D_1 = 1$.
3. If he loses, he will put the double stake in the second round, hence $C_2 = 2$, in the case of $D_1 = -1$.
4. If he wins in the second round, he will pay no more starting from the third round.

Otherwise, he will pay his stake doubled again in the third round and so on.

So, we get as a strategy: $C_n = \begin{cases} 0\ ; if\ there\ is\ i \in \{1,2,\ldots,n-1\}\ with\ D_i = 1 \\ 2^{n-1}\ ; \quad otherwise \end{cases}$

You note that C_n depends only on $D_1, D_2, \dots \dots \dots \dots .., D_{n-1}$, hence, it is measurable in respect to $\sigma(D_1, D_2, \dots \dots \dots \dots .., D_{n-1})$.

1.2. **Example: Binary model**:

1.3. **Definition 2**: A stochastic process $X_0, X_1, \dots \dots \dots .., X_T$ is called binary model, if there are random variables: $D_1, D_2, \dots \dots \dots \dots .., D_T$ with values in $\{-1,1\}$ and functions :

$f_n: \mathbb{R}^{n-1} \times \{-1,1\} \to \mathbb{R}$ for $n = 1,2, \dots \dots \dots \dots \dots, T$. ($T \in \mathbb{N}$ a fixed point of time), such that: $X_0 = x_0$, and $X_n = f_n(X_1, X_2, \dots \dots ., X_{n-1}, D_n)$ for every $n = 1,2, \dots \dots \dots \dots \dots, T$ with $\mathcal{F} = \sigma(X)$. We denote the filtration, which is generated by the process $X = (X_0, X_1, \dots \dots \dots .., X_T)$ by $\mathcal{F} = \sigma(X)$. We note that X_n depends only on $X_1, X_2, \dots \dots ., X_{n-1}$ and D_n and does not depend on the full information of the values $D_1, D_2, \dots \dots \dots \dots .., D_n$.

The Petersburg game and the binary model are two examples of a predictable process.

Definition 3: Previsible (Predictable) process:

A stochastic process $C = (C_n; n \in \mathbb{N})$ is called predictable with respect tot he filtration $\mathcal{F} = (\mathcal{F}_n; n \in \mathbb{N})$; if C_0 is a constant and for every $n \in \mathbb{N}$ holds: C_n is $\mathcal{F}_{n-1}$ −measurable.[1]

Discussion and results:

To the petersburg game: We continue the last example. We put:

$X_n := D_1 + D_2 + \cdots \dots \dots \dots .. + D_n$ for $n \in \mathbb{N}$, then $X = (X_0, X_1, \dots \dots .)$ or

$X = (X_n)_{n \in \mathbb{N}}$ is a martingale.

The strategy of the game: $C_n = \begin{cases} 2^{n-1}\,; if\ D_1 = D_2 = \cdots = D_{n-1} = -1 \\ 0 \quad ; \qquad otherwise \end{cases}$

and $C_0 = 1$ is predictable and local bounded.

Suppose that $S = (S_0, S_1, \dots \dots \dots)$ or $[\,S = (S_n)_{n \in \mathbb{N}}]$ is a sequence of random variables such that: $S_n := \sum_{i=1}^{n} C_i . D_i = (C.X)_n$ is the gaining after n rounds.

Then $S = (S_0, S_1, \dots \dots \dots)$ is a martingale. [1]

But, what is a martingale?

Definition 4: A process $X = (X_n)_{n \in \mathbb{N}}$ is called a martingale with respect to the natural filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ $(\mathcal{F}_n = \sigma(X_0, X_1, \dots \dots \dots, X_n)\ and\ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F})$

i. If X is $\mathcal{F}$ −adapted

ii. $E(|X_n|) < \infty\ ;\ \forall n \in \mathbb{N}$

iii. $E\left(X_n \,\middle|\, \mathcal{F}_{n-1}\right) = X_{n-1}\,; almost\ surely\ for\ every\ n \in \mathbb{N}^*$

Now, S which is mentioned above is an example for our topic (The Discrete Stochastic Integration).[1],[3]

Definition 5: Suppose that $X = (X_n)_{n \in \mathbb{N}}$ is a real, $\mathcal{F}$ −adapted process and $(C_n)_{n \in \mathbb{N}}$ is real-valued und $\mathcal{F}$ −predictable.

We define the stochastic process $C.X$ as: $(C.X)_n := \sum_{m=1}^{n} C_m.(X_m - X_{m-1})$ for $n \in \mathbb{N}$ and call the process $C.X$ the discrete stochastic integral of C with respect to X.

If X is a martingale we will call $C.X$ the martingale transform of Xtoo.[1],[3],[5]

Remark: Under the stochastic integral $\int C.dX$ from C to X we understand the discrete martingale $C.X$, which is defined as above:

$$(C.X)_n = \sum_{m=1}^{n} C_m.(X_m - X_{m-1}) = \sum_{m=1}^{n} C_m.(\Delta_m X)\text{"} = \int_0^n \underbrace{C}_{Integrand} . \underbrace{dX}_{Integrator} \text{"}$$

(as analogue) such that $\Delta_m X = X_m - X_{m-1}$. Martingales act here as good integrators for a discrete stochastic integration. [3]

Properties of the DSI:

Theorem 1: (A fundamental principle : You can't beat the system!)

If X is a martingale and C local bounded (it means, every C_n is bounded, $\exists K_n \in [0,\infty[$ $constant\ with\ |C_n(\omega)| \leq K_n;\ \forall\omega$), then $C.X$ is a martingale.

Proof: We must show the following three conditions:

i. $(C.X)_n$ is $\mathcal{F}_n$ −measurable, it means $(C.X)$ is $\mathcal{F}$ −adapted.

ii. $C.X$ is in L^1, it means integrable.

iii. $E\big((C.X)_{n+1}\,\big|\,\mathcal{F}_n\big) = (C.X)_n$ almost surely for every $n \in \mathbb{N}$.

i : We want to show that if X is $\mathcal{F}$ −measurable, then $C.X$ will be $\mathcal{F}$ −measurable too. $(C.X)_n = \sum_{m=1}^{n} C_m(X_m - X_{m-1})$ is a function of $X_0, X_1, \dots, X_n$ and therefore $(C.X)_n$ is $\mathcal{F}_{n-1}$ −measurable.

ii : $(C.X)_n = \sum_{m=1}^{n} \underbrace{\underbrace{C_m}_{bounded} . \underbrace{(X_m - X_{m-1})}_{integrable}}_{this\ remains\ integrable}$ with:

$E(|(C.X)_n|) \leq E(\sum_{m=1}^{n}|C_m|.|X_m - X_{m-1}|) \leq \sup_m(K_m).2.\sum_{m=1}^{n} E(|X_m|) < \infty$, hence: $C.X \in L^1$.

iii : We want to show that: $E\big((C.X)_{n+1}\,\big|\,\mathcal{F}_n\big) = (C.X)_n$ and for this task it is enough to write: $E\big((C.X)_{n+1} - (C.X)_n\,\big|\,\mathcal{F}_n\big) =$

$$E\left(\sum_{m=1}^{n+1} C_m(X_m - X_{m-1}) - \sum_{m=1}^{n} C_m(X_m - X_{m-1})\,\Big|\,\mathcal{F}_n\right) =$$

$$E\big(C_{n+1}(X_{n+1} - X_n)\,\big|\,\mathcal{F}_n\big) = C_{n+1}.\ \underbrace{E\Big((X_{n+1} - X_n)\,\Big|\,\mathcal{F}_n\Big)}_{=0\ because\ X\ is\ a\ martingale} = 0,\ [3]$$

Theorem 2: Theorem of stability for a stochastic integral

Suppose that $(X_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}$ −adapted, real stochastic process with: $E(|X_0|) < \infty$, then:

X is a martingale if and only if the stochastic integral $C.X$ is a martingale, for every local bounded predictable process C.

Proof: For the proof, it is enough to show that if $C.X$ is a martingale, then X will be a martingale too. It means:We have to show that: $E\big(X_{n+1}\,\big|\,\mathcal{F}_n\big) = X_n$.

Either $C_{n+1} = 0;\ \forall n \in \mathbb{N}$, it means that $(C_n)_{n\in\mathbb{N}}$ is a deterministic process and this process is not interesting here. Or $E\Big((X_{n+1} - X_n)\,\Big|\,\mathcal{F}_n\Big) = 0 \Rightarrow E\Big((X_{n+1})\,\Big|\,\mathcal{F}_n\Big) = E(X_n|\mathcal{F}_n) = X_n$, it means X is a martingale. [1]

Definition 6: The Covariation:

Suppose that M_1, M_2 two martingales, then $[M_1, M_2]_n$ is called the covariation of M_1 and M_2, such that: $[M_1, M_2]_n := \sum_{m=1}^{n} \Delta_m(M_1).\Delta_m(M_2)$.

Theorem 3:

Suppose that M_1, M_2 are two real-valued discrete martingales and C, C' are two predictable real-valued processes then:

i. $C.(C'.M_1) = (C.C').M_1$

ii. $[C.M_1, C'.M_2]_n := (C.C').[M_1, M_2]_n$

Proof: i : LS:=$C.(C'.M_1)_n = \sum_{m=1}^{n} C_m.\Delta_m(C'.M_1) = \sum_{m=1}^{n} C_m.[(C'.M_1)_m - (C'.M_1)_{m-1}] = \sum_{m=1}^{n} C_m.(C'_m.(M_1)_m) = \sum_{m=1}^{n} C_m.C'_m(M_1)_m = (C.C').(M_1)_n$=:RS

ii we write: $[C.M_1, C'.M_2]_n = \sum_{m=1}^{n} \Delta_m(C.M_1).\Delta_m(C'.M_2) = \sum_{m=1}^{n} C_m.\Delta_m(M_1).C'_m.\Delta_m(M_2) =$

$$= \sum_{m=1}^{n} C_m.C'_m.\Delta_m(M_1).\Delta_m(M_2) = \sum_{m=1}^{n} (C.C')_m \Delta_m(M_1).\Delta_m(M_2) =$$

$$= \sum_{m=1}^{n} (C.C')_m .\Delta_m[M_1, M_2] = (C.C').[M_1, M_2]_n$$

Theorem 4: The discrete stochastic integral $Y_n = (C.X)_n$ with a martingale X and a predictable process C is centered and $Var(Y_n) = \sum_{m=1}^{n} E((C_m)^2.(X_m - X_{m-1})^2)$.

Proof:

$E(Y_n) = E(\sum_{m=1}^{n} C_m.(X_m - X_{m-1})) =$

$\sum_{m=1}^{n} E(C_m.\Delta_m X) = \sum_{m=1}^{n} E\left(E\left(C_m.\Delta_m X \,\middle|\, \mathcal{F}_{m-1}\right)\right) = \sum_{m=1}^{n} E(C_m.0) = 0$, because X is a martingale.

$$Var(Y_n) = Var\left(\sum_{m=1}^{n} C_m.(X_m - X_{m-1})\right) = E\left(\left(Y_n - E(Y_n)\right)^2\right) = E((Y_n - 0)^2) =$$

$$= E\left(\left(\sum_{m=1}^{n} C_m.(X_m - X_{m-1})\right)^2\right) =$$

$$E\left(\sum_{m=1}^{n} (C_m)^2 .(X_m - X_{m-1})^2 + \sum_{m \neq k} C_m.C_k.(X_m - X_{m-1}).(X_k - X_{k-1})\right)$$

$$= E\left(\sum_{m=1}^{n} (C_m)^2 .(X_m - X_{m-1})^2 + 2.\sum\sum_{1 \le m < k \le n} C_m.C_k.(X_m - X_{m-1}).(X_k - X_{k-1})\right)$$

Now, we can take the following decomposition:

$$0 < m - 1 < m < k - 1 < k \,; m, k \in \mathbb{N}$$

and suppose that: $I_k = \sum_{m=1}^{k-1} C_m.C_k.(X_m - X_{m-1}).(X_k - X_{k-1}) \implies$

$$Var(Y_n) = E\left(\sum_{m=1}^{n} (C_m)^2 .(X_m - X_{m-1})^2\right) + 2.\sum_{k=2}^{m} E(I_k) \ldots\ldots\ldots.**$$

$$E(I_k) = E\left(E\left(I_k \,\middle|\, \mathcal{F}_{k-1}\right)\right) = \sum_{m=1}^{k-1} E\left(E\left(C_m.C_k.(X_m - X_{m-1}).(X_k - X_{k-1}) \,\middle|\, \mathcal{F}_{k-1}\right)\right)$$

$$= \sum_{m=1}^{k-1} E\left(C_m.C_k.(X_m - X_{m-1}).\underbrace{E\left((X_k - X_{k-1}) \,\middle|\, \mathcal{F}_{k-1}\right)}_{=0}\right) = 0$$

$$Var(Y_n) = E\left(\sum_{m=1}^{n} (C_m)^2 .(X_m - X_{m-1})^2\right)$$

Remark: If we have a simple symmetrical random walk, then it holds:

$(X_m - X_{m-1})^2 = 1 \implies$

$Var(Y_n) = E(\sum_{m=1}^{n} (C_m)^2) = \sum_{m=1}^{n} E(C_m^2)$.

3. The discrete Itô-formula:

3.1. Doob-decomposition and quadratic variation:

Suppose that $X=(X_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}$ −adapted process with $E(|X_n|)<\infty$ for every $n\in\mathbb{N}$.

We want to decompose X into a sum of a martingale and a predictable process. In addition to, we define M_n for every $n\in\mathbb{N}$ as the following:

$M_n := X_0+\sum_{k=1}^{n}\left(X_k-E\left(X_k\,\middle|\,\mathcal{F}_{k-1}\right)\right)$,

$C_n := \sum_{k=1}^{n}\left(E\left(X_k\,\middle|\,\mathcal{F}_{k-1}\right)-X_{k-1}\right)$. It is obviously that $X_n=M_n+C_n$ by construction such that C is predictable with $C_0=0$ and M is a martingale because:

$E\left(M_n-M_{n-1}\,\middle|\,\mathcal{F}_{n-1}\right)=E\left(X_n-E\left(X_n\,\middle|\,\mathcal{F}_{n-1}\right)\,\middle|\,\mathcal{F}_{n-1}\right)=E\left(X_n\,\middle|\,\mathcal{F}_{n-1}\right)-E\left(E\left(X_n\,\middle|\,\mathcal{F}_{n-1}\right)\,\middle|\,\mathcal{F}_{n-1}\right)=E\left(X_n\,\middle|\,\mathcal{F}_{n-1}\right)-E\left(X_n\,\middle|\,\mathcal{F}_{n-1}\right)=0$, [3]

Theorem 5: (Doob-decomposition):

Suppose that $X=(X_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}$ −adapted, integrable process. Then, it exists a unique decomposition $X=M+C$, such that C is predictable with $C_0=0$ and M is a martingale.

This representation of X is called Doob-decomposition.[1],[5].

Definition 7: Suppose that $X=(X_n)_{n\in\mathbb{N}}$ is a quadratic integrable $\mathcal{F}$ −martingale. The unique specific predictable process C, from which we get the martingale $(X_n^2-C_n)_{n\in\mathbb{N}}$, is called the quadratic variation process of X and is denoted by the formula: $(\langle X\rangle_n)_{n\in\mathbb{N}} := C$. [1].

Example: Suppose that $X_1, X_2, \dots\dots\dots$ are independent, quadratic integrable centered random variables. Then a quadratic integrable martingale is defined by: $M_n=X_1+\cdots+X_n$ with the quadratic variation: $\langle M\rangle_n=\sum_{i=1}^{n}E(X_i^2)$, then: $C_n=\sum_{i=1}^{n}E\left(X_i^2\,\middle|\,X_1,X_2,\dots,X_{i-1}\right)=\sum_{i=1}^{n}E(X_i^2)$. We note that, it is not sufficient for this simple representation of $\langle M\rangle$, that $X_1, X_2, \dots\dots\dots$ are uncorrelated.[1]

3.2. The discrete Itô-formula:

Example: Suppose that $(X_n)_{n\in\mathbb{N}}$ is the one-dimensional symmetrical simple random walk, $X_n := \sum_{i=1}^{n}R_i$ for every $n\in\mathbb{N}$, such that: $(R_i)_{i\in\mathbb{N}}$ independent identically distributed random variables with: $P(R_i=1)=1-P(R_i=-1)=\frac{1}{2}$.

Obviously, X is a martingale, hence $|X|$ is a submartingale $\left(X_n\le E\left(X_{n+1}\,\middle|\,\mathcal{F}_n\right)\right); n\in\mathbb{N}$

Suppose that $|X|=M+C$ the Doob-decomposition of $|X|$, then is:

$C_n=\sum_{i=1}^{n}\left(E\left(|X_i|\,\middle|\,\mathcal{F}_{i-1}\right)-|X_{i-1}|\right)$. Now, $|X_i|=\begin{cases}|X_{i-1}|+R_i\,; if\ X_{i-1}>0\\ |X_{i-1}|-R_i\,; X_{i-1}<0\\ 1\quad ;\ X_{i-1}=0\end{cases}$

We want to generalize this example. Obviously, we need (except in the last formula), that X is a random walk, but the important thing here that the difference

$\Delta_n X := X_n-X_{n-1}$ can only take the values $-1,1$.

Now, suppose that X is a martingale with $|X_n-X_{n-1}|=1$ almost surely for every $n\in\mathbb{N}$ and with $X_0=x_0\in\mathbb{Z}$ almost surely and suppose that $f:\mathbb{Z}\to\mathbb{R}$ is an arbitrary function, then $Y := \left(f(X_n)\right)_{n\in\mathbb{N}}$ is an integrable adapted process, because it holds:

For every $n \in \mathbb{N}$ $|f(X_n)| \leq \max_{x \in \{x_0-n,\ldots\ldots,x_0+n\}} |f(x)|$. To identify the Doob-decomposition of Y, we have to define the first and the second discrete derivation of f. According to the main theorem of differential and integral calculus we can write:

$f(x) - f(0) = \int_0^x f'(y).dy$ [1],[2],but what happens when X is a martingale?

Theorem 6: Suppose that X is a one-dimensional symmetrical random walk and is defined as in the last example. Suppose that: $f'(x) := \frac{f(x+1)-f(x-1)}{2}$; $x \in \mathbb{z}$ and

$f''(x) := f(x+1) + f(x-1) - 2.f(x)$; $x \in \mathbb{z}$. We put more $F_n' := f'(X_{n-1})$ and $F_n'' := f''(X_{n-1})$, then the Itô-formula is:

$$f(X_n) = f(X_0) + (F'.X)_n + \frac{1}{2}\,(F''.\langle X\rangle)_n; n \in \mathbb{N}$$

Proof: By distinguishing between the situations $X_n = X_{n-1} - 1$ and $X_n = X_{n-1} + 1$, we see that for every $n \in \mathbb{N}$: $f(X_n) - f(X_{n-1}) = \frac{f(X_{n-1}+1)-f(X_{n-1}-1)}{2}.(X_n - X_{n-1}) + \frac{1}{2}.f(X_{n-1}-1) + \frac{1}{2}.f(X_{n-1}+1) - f(X_{n-1}) = f'(X_{n-1}).(X_n - X_{n-1}) + \frac{1}{2}.f''(X_{n-1})$

$= F_n'.(X_n - X_{n-1}) + \frac{1}{2}.F_n''.(X_n - X_{n-1})^2$. In total we get the discrete Itô-formula:

$$f(X_n) = f(x_0) + \sum_{i=1}^{n} f'(X_{i-1}).(X_i - X_{i-1}) + \sum_{i=1}^{n} \frac{1}{2}.f''(X_{i-1}) =$$

$$= f(x_0) + (F'.X)_n + \sum_{i=1}^{n} \frac{1}{2}.F_i'' = f(x_0) + \underbrace{(F'.X)_n}_{DSI} + \frac{1}{2}.(F''.\langle X\rangle)_n \quad [2]$$

Remark: $M_n = f(x_0) + (F'.X)_n$ is a martingale, because $F'$is predictable

$$\left(|F_n'| \leq \max_{x \in \{x_0-n,\ldots\ldots,x_0+n\}} f'(x)\right)$$

and $C_n := \sum_{i=1}^{n} \frac{1}{2}.F_i''$; $n \in \mathbb{N}$ is predictable. Hence, $f(x) := \big(f(x_n)\big)_{n\in\mathbb{N}} = M + C$ and that is exactly the Doob-decomposition of $f(x)$.

Example: $f(x) = x^2$. Here is: $f'(x) = \frac{f(x+1)-f(x-1)}{2} = \frac{(x+1)^2-(x-1)^2}{2} = 2x$ and

$f''(x) = (x-1)^2 + (x+1)^2 - 2x^2 = 2, x_0 = 2 \Longrightarrow f(x_0) = f(0) = 0$

$$\langle X\rangle_n := \sum_{m=1}^{n} (\Delta_m X)^2 = \sum_{m=1}^{n} 1 = n$$

Suppose that $X_n = \sum_{i=1}^{n} R_i$. We know already, that the sum of independent centered random variables is a martingale, $x_0 = 0, f(X_n) =$?

$$f(X_n) = 0 + \sum_{i=1}^{n} f'(X_{i-1}).(X_i - X_{i-1}) + \sum_{i=1}^{n} \frac{1}{2}.f''(x_{i-1}) =$$

$$= \sum_{i=1}^{n} 2.X_{i-1}.R_i + \sum_{i=1}^{n} \frac{1}{2}.2 = \underbrace{2.\sum_{i=1}^{n}\left(\sum_{j=1}^{i-1} R_j\right).R_i + n}_{DSI=\left(\sum_{i=1}^{n} R_i\right)^2}$$

because $\sum_{j=1}^{i-1} R_j$ is $\mathcal{F}_{i-1}$ −measurable.

Conclusions and recommendations:

This research enables us to present martingales as good integrators of a discrete stochastic integration and presents five important properties of the DSI through five theorems, which are precisely proved. But the question now is:

What happens when our process is continuous?

Can we say the same things about stochastic integral like $C.X$ when $X$is a continuous martingale.

I hope that you can continue the study in continuous times.

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