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Augmented G_graded Modules

Dr.Mashhour AL-Refaee*

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□ ABSTRACT □

Let G be a multiplicative group with identity e and R be an associative G-graded ring with unity 1. Let R_e be the identity component of R and R_e-gr be the category of all graded R_e-modules and their graded R_e-maps. In this paper we study the augmented G-graded modules and give some of their properties. We define R-Agr (the category of all augmented G-graded R-modules and their augmented G-graded R-maps), and show there is a functor $(\ )_e$ from R-Agr to R_e-gr and a functor (-) from R_e-gr to R-Agr. Moreover, $(\ )_e(-)$ is equivalent to $1_{R_e\text{-gr}}$ and the two functors are equivalent for

*faculty of sciences , department of the math , Yarmouk university , Erbed , Jordan

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الموديلات المدرجة الزائدة

الدكتور مشهور الرفاعي*

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□ الملخّص □

لتكن G *زمرة ضربية عنصر وحدتها* E *ولتكن* R *حلقة تجميعية عنصر وحدتها* 1 *ومدرجة حسب الزمرة* G *أي أنه يوجد أسرة* $\{R_g : g \in G\}$ *من الزمرة الجزئية الجمعية من الحلقة* R *بحيث يكون* $R = \bigoplus_{g \in G} R_g$ *و* $R_g R_h \subseteq R_{gh}$ *وذلك مهما يكن* H,G *من الزمرة* G . *لنفرض أن* R_e *هي المركبة المحايدة للحلقة المدرجة* R . *و* $R_e - gr$ *هي طائفة جميع الموديلات المدرجة حسب* R_e *بالإضافة إلى تطبيقاتها المدرجة .*

الغاية من هذا البحث هو دراسة مفهوم جديد هو الموديلات الزائدة وإعطاء بعض خصائصه. لقد قمنا بتعريف R-Agr *والتي تمثل طائفة جميع الموديلات المدرجة الزائدة وتطبيقاتها المدرجة . وقد أثبتت الدراسة وجود دلال* e( ) (function) *من الطائفة* R-Agr *إلى الطائفة* R_e gr - *وأيضا وجود دلال* (–) *من الطائفة* R_e gr - *إلى الطائفة* R-Agr . *كما قمنا بإثبات وجود علاقة تكافؤ بين* (–) e( ) *و* 1_{Re-gr} . *وأيضا يكون الدلال الأول* e( ) *مكافئ للدلال الثاني* (–) *في الحالة التي تكون فيها دعامة الحلقة تساوي الزمرة* G .

*كلية العلوم – قسم الرياضيات – جامعة اليرموك – إربد – الأردن.

0. Introduction.

Let G be a multiplicative group with identity e and R be an associative G-graded ring with unity 1. Let R_e be the identity component of R. In [5] we studied the G-graded rings for which the identity component R_e is itself a G-graded subring satisfying some related conditions with the graduation of G. We called these rings augmented G-graded rings.

In this paper we study the augmented G-graded modules and give some of their properties. It is well known that if R is a strongly G-graded ring then there is an equivalence from R-gr (the category of all G-graded R-modules and their graded R-maps) to R_e-mod (the category of all R_e-modules and their R_e-maps). In this paper we define a new category R-Agr and show that there is a functor $(\ )_e$ from R-Agr to R_e-gr and a functor $(\overline{-})$ from R_e-gr to R-Agr. Moreover, $(\ )_e(\overline{-})$ is equivalent to $1_{R_e\text{-gr}}$. Also, we show that if $\mathrm{supp}(R,G) = \left\{g \in G : R_g \neq 0\right\} = G$ then $(\overline{-})(\ )_e$ is equivalent to $1_{R\text{-Agr}}$, i.e., the two functors are equivalent.

1. Preliminaries.

In this section we give some basic definitions and facts which will be used later on.

Definition 1.1. Let G be a group with identity e. Then a ring R is G-graded if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all g, h ∈ G. We denote for this graduation by (R,G) and R_e will be the identity component of R.

We say (R,G) is strongly graded ring if $R_g R_h = R_{gh}$ for all g, h ∈ G. We consider $\text{supp}(R,G) = \left\{g \in G: R_g \neq 0\right\}$.

Definition 1.2. Let R be a G-graded ring and M is a (left) R-module. Then M is a (left) graded R-module if there exist additive subgroups M_g of M such that $M = \bigoplus_{g \in G} M_g$ and $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all σ, τ ∈ G. A map f: M ⟶ M is a graded R-map if f is R-linear and $f(M_g) \subseteq M_g$ for all g ∈ G.

We consider R-gr to be the category of all left graded R-modules and their graded R-maps and R_e-mod to be the category of all R_e-modules and their R_e-maps. It is well known that R-gr is a Grothendieck category [3]. The connection between the category R-gr and R_e-mod is given by the following:

Theorem 1.3 (Theorem 2.8 of [1]). Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a strongly graded ring. Then the functor

$R\otimes_{R_e} \cdot : R_e\text{-mod} \longrightarrow$ R-gr given by

$M \longrightarrow R\otimes_{R_e} M$, where $M \in R_e$-mod and $R\otimes_{R_e} M$ is a graded R-module by the grading $\left(R\otimes_{R_e} M\right)_\sigma = R_\sigma \otimes_{R_e} M$, is an equivalence. Its inverse is the functor

$(\cdot)_e$: R-gr $\longrightarrow R_e$-mod given by

$M \longrightarrow M_e$ where $M \in$ R-gr and $M = \bigoplus_{\sigma\in G} M_\sigma$.

Theorem 1.4 (Proposition 1.2 of [4]). Let R be a G-graded ring. Then R is strongly graded ring iff every $M \in$ R-gr is strongly graded module.

Definition 1.5 ([5]). A ring R is said to be an augmented G-graded ring if it satisfies the following conditions:

(1) $R = \bigoplus_{g\in G} R_g$ is a G-graded ring.

(2) If R_e is the identity component of the graduation given in (1) then $R_e = \bigoplus_{g\in G} R_{e-g}$ where R_{e-g} is an additive subgroup of R_e and $R_{e-g} R_{e-h} \subseteq R_{e-gh}$ for all $g, h \in G$. (R_e is a G-graded ring).

(3) For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = \bigoplus_{h\in G} R_{e-h} r_g$, we assume $r_e = 1$.

(4) If $g, h \in G$ and r_g, r_h are both non-zero, then $r_g r_h = r_{gh}$ and $(xr_g)(yr_h) = xy\, r_{gh}$ for all $x, y \in R_e$.

Lemma 1.6 (Proposition 2.6 of [5]). Let (R,G) be augmented G-graded ring and $supp(R,G) = G$. Then R is a strongly G-graded ring.

2. Augmented graded modules.

In this section we define the augmented G-graded modules. Then we give some examples and properties of these modules.

Definition 2.1. Let R be an augmented G-graded ring. An R-module M is said to be an augmented G-graded R-module if it satisfies the following conditions:

1. $M = \bigoplus_{g\in G} M_g$ where M_g is an R_e-submodule of M such that $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ (i.e., $M \in R\text{-gr}$).
2. $M_g = \bigoplus_{h\in G} M_{g-h}$ where M_{g-h} is an R_{e-e}-submodule of M_g and $R_{e-\sigma} M_{g-h} \subseteq M_{g-\sigma h}$ for all $\sigma, g, h \in G$ (i.e., $M_g \in R_e\text{-gr}$).

3. $R_{g-h} M_{\sigma-\tau} \subseteq M_{g\sigma-h\tau}$ for all $g, h, \sigma, \tau \in G$.

Let M, N be augmented G-graded R-modules. An R-map $f: M \longrightarrow N$ is said to be an augmented G-graded R-map if $f(M_{g-h}) \subseteq N_{g-h}$ for all $g, h \in G$. We consider R-Agr to be the category of all augmented G-graded R-modules and their augmented G-graded R-maps.

Remarks 2.2. (1) If R is an augmented G-graded ring, supp(R,G)=G and M∈R-Agr then $M_{g-h} = r_g M_{e-h}$ for all $g, h \in G$.

Clearly $r_g M_{e-h} \subseteq R_{g-e} M_{e-h} \subseteq M_{g-h}$. Let $x \in M_{g-h}$. By Lemma 1.6, (R,G) is strongly graded ring and hence by Theorem 1.4, M is strongly graded module. Therefore, $M_{g-h} \subseteq M_g = r_g M_e$ and hence $x = r_g m$ for some $m \in M_e = \oplus_{h\in G} M_{e-h}$. Assume $m = \sum_{j=1}^{n} m_{e-h_j}$, then $x = r_g m = r_g m_{e-h_1} \oplus \dots \oplus r_g m_{e-h_n}$. Since $M_g = \oplus_{h\in G} M_{g-h}$, $n = 1$ and $h_1 = h$, i.e., $M_{g-h} \subseteq r_g M_{e-h}$.

(2) If $M \in$ R-Agr then $M \in R_e$-gr with $M_{(g)} = \oplus_{\sigma\in G} M_{\sigma-g}$

1. $M \in R_e$-mod (with the same product of R on M).
2. $M_{(g)} \in R_{e-e}$-mod because $R_{e-e} M_{\sigma-g} \subseteq M_{\sigma-g}$ for all $\sigma \in G$.
3. $R_{e-\sigma} M_{(g)} \subseteq M_{(\sigma g)}$ because $R_{e-\sigma} M_{\tau-g} \subseteq M_{\tau-\sigma g} \subseteq M_{(\sigma g)}$ for all $\tau \in G$.
4. Clearly $M = \oplus_{g\in G} M_{(g)}$.

Definition 2.3. Let M be an augmented G-graded R-module, and X be an R-submodule of M. then X is said to be an augmented G-graded R-submodule of M if it satisfies the following conditions:

1. $X = \oplus_{g \in G} X_g$ where $X_g = X \cap M_g$, i.e., X is a G-graded R-submodule of M.
2. $X_g = \oplus_{\sigma \in G} X_{g-\sigma}$ where $X_{g-\sigma} = X_g \cap M_{g-\sigma}$.

Remarks 2.4. Let $M \in$ R-Agr and X be an augmented G-graded R-submodule of M. Then

1. X_g is R_e-submodule of X.

   Since $R_e X_g \subseteq X$ and $R_e M_g \subseteq M_g$ we have $R_e X_g \subseteq X \cap M_g = X_g$.
2. $R_{e-\sigma} X_{g-\tau} \subseteq X_{g-\sigma\tau}$.

   Since $R_{e-\sigma} X_{g-\tau} \subseteq X_g$ and $R_{e-\sigma} X_{g-\tau} \subseteq R_{e-\sigma} M_{g-\tau} \subseteq M_{g-\sigma\tau}$ we have $R_{e-\sigma} X_{g-\tau} \subseteq X_g \cap M_{g-\sigma\tau} = X_{g-\sigma\tau}$
3. $R_{\sigma-\tau} X_{g-h} \subseteq X_{\sigma g-\tau h}$.

   Since $R_{\sigma-\tau} X_{g-h} \subseteq R_\sigma X_g \subseteq X_{\sigma g}$ and $R_{\sigma-\tau} X_{g-h} \subseteq R_{\sigma-\tau} M_{g-h} \subseteq M_{\sigma g-\tau h}$ we have $R_{\sigma-\tau} X_{g-h} \subseteq X_{\sigma g} \cap M_{\sigma g-\tau h} = X_{\sigma g-\tau h}$.
4. $R_\sigma X_g \subseteq X_{\sigma g}$.

   Since $R_\sigma X_g \subseteq X$ and $R_\sigma X_g \subseteq M_{\sigma g}$ we have $R_\sigma X_g \subseteq X \cap M_{\sigma g} = X_{\sigma g}$.

Proposition 2.5. Let $M \in$ R-Agr and X be a G-graded R_e-submodule of M_g. Then RX is an augmented G-graded R-submodule of M.

Proof: (1) RX is an R-submodule of M.

(2) Since $(RX)_\sigma = RX \cap M_\sigma = R_{\sigma g^{-1}} X$ we have $RX = \underset{\sigma\in G}{\oplus} (RX)_\sigma$.

(3) Since $(RX)_{\sigma-\tau} = (RX)_\sigma \cap M_{\sigma-\tau} = \underset{h\in G}{\oplus} R_{\sigma^{-1}g-h} X_{h^{-1}\tau}$ we have

$(RX)_\sigma = \underset{\tau\in G}{\oplus} (RX)_{\sigma-\tau}$.

Now we give two essential examples of augmented graded modules. The idea of those examples is similar to the examples of augmented graded rings given in [5].

<u>Example 2.6.</u> Let R be a G-graded ring and $M \in R\text{-gr}$. Let R[G] be the group ring of R over G. Then clearly R[G] is an augmented G-graded ring with:

$R[G]_g = R.g$, $R[G]_{e-g} = R_g.e$ and $r_g = 1.g$ for all $g \in G$. Let $M[G] = \underset{g\in G}{\oplus} M.g$. For $r.\sigma \in R[G]$ and $m.\tau \in M[G]$, define $(r\sigma)(m\tau) = rm\sigma\tau$. Then $M[G] \in R[G]\text{-Agr}$ with $M[G]_g = M.g$ and $M[G]_{g-\sigma} = M_\sigma.g$.

1. Since $M[G]_g$ is $R[G]_e$-submodule of M[G] and $R[G]_g\, M[G]_\sigma = R.g\, M.\sigma \subseteq M.g\sigma = M[G]_{g\sigma}$ we have $M[G] = \underset{g\in G}{\oplus} M[G]_g$.
2. Since $M[G]_{g-h}$ is R_{e-e}-submodule of $M[G]_g$ and $R[G]_{e-\sigma}\, M[G]_{g-h} = R_\sigma.e\, M_h.g \subseteq M_{\sigma h}.g = M[G]_{g-\sigma h}$ we have $M[G]_g = \underset{h\in G}{\oplus} M[G]_{g-h}$, i.e., $M[G]_g$ is a G-graded R_e-module.
3. $R[G]_{g-h}\, M[G]_{\sigma-\tau} = R_h.g\, M_\tau.\sigma \subseteq M_{h\tau}g\sigma = M[G]_{g\sigma-h\tau}$ for all $g, h, \sigma, \tau \in G$.

Example 2.7. Let R be a G-graded ring and $M \in R\text{-gr}$. Let $\overline{R[G]}$ be the left free R-module with basis G. For the elements $\lambda_\sigma \tau$, $\lambda_{\sigma'} \tau'$ where $\lambda_\sigma \in R_\sigma$ and $\lambda_{\sigma'} \in R_{\sigma'}$ define $(\lambda_\sigma \tau)\left(\lambda_{\sigma'} \tau'\right) = \left(\lambda_\sigma \lambda_{\sigma'}\right) \sigma'^{-1} \tau \sigma' \tau'$. With this product $\overline{R[G]}$ is an associative ring with unity 1.e, (see [2]). One can easily show $\overline{R[G]}$ is an augmented G-graded ring with:

$\overline{R[G]}_g = \bigoplus_{\sigma \in G} R_{g\sigma^{-1}} \sigma$, $\overline{R[G]}_{e-g} = R_g \, g^{-1}$ and $r_g = 1.g$ for all $g \in G$.

Let $\overline{M[G]} = \bigoplus_{g \in G} M.g$. Then $\overline{M[G]}$ is an augmented G-graded $\overline{R[G]}$-module with:

$(\lambda_\sigma \tau)\left(m_{\sigma'} , \tau'\right) = \lambda_\sigma m_{\sigma'} \, \sigma'^{-1} \tau \sigma' \tau'$ where $\lambda_\sigma \in R_\sigma$, $m_{\sigma'} \in M_{\sigma'}$,

$\overline{M[G]}_g = \bigoplus_{\sigma \in G} M_{g\sigma^{-1}} \sigma$ and $\overline{M[G]}_{g-h} = M_h \, h^{-1} g$.

1. $\overline{M[G]} \in \overline{R[G]}$-mod (see [2]).
2. $\overline{M[G]}_g = \bigoplus_{\sigma \in G} M_{g\sigma^{-1}} \sigma$ is $\overline{R[G]}_e$-submodule of $\overline{M[G]}$ because $R_h \, h^{-1} M_{g\sigma^{-1}} \sigma \le M_{hg\sigma^{-1}} \sigma g^{-1} h^{-1} g \le \overline{M[G]}_g$. Since $R_{g\sigma^{-1}} \sigma M_{h\tau^{-1}} \tau \le M_{g\sigma^{-1} h\tau^{-1}} \tau h^{-1} \sigma h \le \overline{M[G]}_{gh}$ we have $\overline{R[G]}_g \overline{M[G]}_h \le \overline{M[G]}_{gh}$.
3. Clearly $\overline{M[G]}_{g-h} = M_h \, h^{-1} g$ is $\overline{R[G]}_{e-e}$-submodule of $\overline{M[G]}_g$ and $\overline{R[G]}_{e-\sigma} \overline{M[G]}_{g-h} = R_\sigma \, \sigma^{-1} M_h \, h^{-1} g \le M_{\sigma h} \, h^{-1} \sigma^{-1} g = \overline{M[G]}_{g-\sigma h}$. Hence $\overline{M[G]}_g = \bigoplus_{h \in G} \overline{M[G]}_{g-h}$, i.e., $\overline{M[G]}_g$ is a G-graded $\overline{R[G]}_e$-module.
4. $\overline{R[G]}_{g-h} \overline{M[G]}_{\sigma-\tau} = R_h \, h^{-1} g \, M_\tau \, \tau^{-1} \sigma \le R_{h\tau} \, \tau^{-1} h^{-1} g \sigma = \overline{M[G]}_{g\sigma - h\tau}$ for all $h, g, \sigma, \tau \in G$.

3. Equivalent functors.

In this section R will be an augmented G-graded ring. Let $M \in R_e$-gr and $\overline{M} = \underset{g \in G}{\oplus} M.g$. For $m \in M$, $\sigma \in G$ and $x_{e-h} \in R_{e-h}$ let $x_{e-h}\ r_g \in R_{e-h} r_g$ and $m\sigma \in \overline{M}$. Define $\left(x_{e-h} r_g\right) m\sigma = x_{e-h}\ mg\sigma$. Then one can easily extend this product to a multiplication of R on $\overline{M}$. With these notations we have the following assertions:

Proposition 3.1. $\overline{M} \in$ R-Agr with $\overline{M}_g = M.g$ and $\overline{M}_{g-h} = M_h.g$.

Proof: (1) To show $\overline{M} \in$ R-mod we only need to show that if $x_{e-h_1} r_{g_1} \in R_{g_1-h_1}-0$, $x_{e-h_2} r_{g_2} \in R_{g_2-h_2}-0$ and $m\sigma \in \overline{M}$ then

$$\left[\left(x_{e-h_1} r_{g_1}\right)\left(x_{e-h_2} r_{g_2}\right)\right](m\sigma) = \left(x_{e-h_1} r_{g_1}\right)\left[\left(x_{e-h_2} r_{g_2}\right)(m\sigma)\right].$$

But $\left[\left(x_{e-h_1} r_{g_1}\right)\left(x_{e-h_2} r_{g_2}\right)\right](m\sigma) = \left[x_{e-h_1}\ x_{e-h_2} r_{g_1 g_2}\right](m\sigma)$

$$= x_{e-h_1}\ x_{e-h_2}\ m g_1 g_2 \sigma = \left(x_{e-h_1} r_{g_1}\right)\left(x_{e-h_2} m g_2 \sigma\right)$$

$$= \left(x_{e-h_1} r_{g_1}\right)\left[\left(x_{e-h_2} r_{g_2}\right) m\sigma\right].$$

(2) For $m \in M$ and $x_{e-h_1} \in R_{e-h_1}$ let $x_{e-h_1} r_g \in R_g$ and $mh \in \overline{M}_h$.

Then $\left(x_{e-h_1} r_g\right)(mh) = x_{e-h_1} mgh \in \overline{M}_{gh}$, i.e., $R_g \overline{M}_h \subseteq \overline{M}_{gh}$.

Clearly $\overline{M}_g$ is R_e-submodule of $\overline{M}$ and $\overline{M} = \underset{g \in G}{\oplus} \overline{M}_g$.

(3) $\overline{M}_g = \underset{h \in G}{\oplus} M_h.g = \underset{h \in G}{\oplus} \overline{M}_{g-h}$ where $\overline{M}_{g-h}$ is R_{e-e}-submodule of $\overline{M}_g$ and $R_{e-h_1}\ \overline{M}_{g-h} = R_{e-h_1}\ M_h.g \subseteq M_{h_1 h}.g = \overline{M}_{g-h_1 h}$.

(4) $R_{g-h}\ \overline{M}_{\sigma-\tau} = \left(R_{e-h} r_g\right)\left(M_\tau \sigma\right) \subseteq M_{h\tau}\ g\sigma = \overline{M}_{g\sigma-h\tau}$ for all g, h, σ, $\tau \in G$.

Proposition 3.2. Let $M, N \in R_e$-gr and $f: M \longrightarrow N$. Then $\overline{f}: \overline{M} \longrightarrow \overline{N}$ given by $\overline{f}(m\sigma) = f(m)\,\sigma$ is a morphism in R-Agr.

Proof: (1) $\overline{f} \in$ R-mod because $\overline{f}(m_1\sigma + m_2\sigma) = \overline{f}\left((m_1 + m_2)\,\sigma\right) = f(m_1 + m_2)\,\sigma = f(m_1)\,\sigma + f(m_2)\sigma$ and $\overline{f}\left(x_{e-h}\, r_g\, m\sigma\right) = \overline{f}\left(x_{e-h}\, mg\sigma\right) = f\left(x_{e-h}\, m\right)g\sigma = x_{e-h}\, f(m)g\sigma = x_{e-h}\, r_g\, f(m)\sigma = x_{e-h}\, r_g\, \overline{f}(m\sigma)$.

(2) $\overline{f}(\overline{M}_{\sigma-\tau}) = \overline{f}(M_\sigma\ \tau) = f(M_\sigma).\tau \in N_\sigma.\tau = \overline{N}_{\sigma-\tau}$.

Proposition 3.3. (1) $\overline{(-)}$: R_e-gr $\longrightarrow$ R-Agr such that $M \longrightarrow \overline{M}$ and $f \longrightarrow \overline{f}$ is a functor.

(2) $(\ )_e$: R-Agr $\longrightarrow R_e$-gr such that $M \longrightarrow M_e$ and $f \longrightarrow f|_{M_e}$ is a functor.

Proof: (1) To show $\overline{(-)}$ is a functor:

1. If $M \in R_e$-gr then $\overline{M} \in$ R-Agr by Proposition 3.1.
2. If $f \in R_e$-gr then $\overline{f} \in$ R-Agr by Proposition 3.2.
3. Let $f: M \longrightarrow N$, $g: N \longrightarrow P$ and $\overline{g\circ f}: \overline{M} \longrightarrow \overline{P}$. Then $\overline{g\circ f}(m\sigma) = (g\circ f(m))\ \sigma = g\ (f(m))\ \sigma = \overline{g}(f(m)\sigma) = \overline{g}\circ\overline{f}(m\sigma)$, i.e., $\overline{g\circ f} = \overline{g}\circ\overline{f}$.
4. Let $1_M: M \longrightarrow M$ be the identity function in R_e-gr. Then clearly $\overline{1}_{\overline{M}}: \overline{M} \longrightarrow \overline{M}$ is the identity function on $\overline{M}$ in R-Agr.

(2) To show $(\ )_e$ is a functor.

Let $f: M \longrightarrow N$ and $g: N \longrightarrow P$ be elements in R-Agr. Then $(g\circ f)|_{M_e}: M_e \longrightarrow P_e$ and $(g\circ f)|_{M_e} = \left(g|_{N_e}\right) \circ \left(f|_{M_e}\right)$.

The other parts are obvious.

:oposition 3.4. $(\ )_e(-)$ is equivalent to $1_{R_e\text{-gr}}$.

:oof: We only need to define a natural transformation

: $(\ )_e(-) \longrightarrow 1_{R_e\text{-gr}}$ in which α_M: $(\overline{M})_e \longrightarrow M$ is an equivalence

ı R_e-gr for every $M \in R_e$-gr.

Let $M \in R_e$-gr. Define α_M: $(\overline{M})_e \longrightarrow M$ by $\alpha_M(m.e) = m$.

ıen clearly α_M is a morphism in R_e-gr. Also, for every

›rphism $f: M \longrightarrow N$ in R_e-gr the following diagram commutes:

$$\begin{array}{ccc} (\overline{M})_e & \xrightarrow{\ \alpha_M\ } & M \\ {\scriptstyle \overline{f}|_{(\overline{M})_e}}\downarrow & & \downarrow{\scriptstyle f} \\ (\overline{N})_e & \xrightarrow[\ \alpha_N\ ]{} & N \end{array}$$

ıerefore, α is a natural transformation. The equivalence of

ı in R_e-gr is obvious.

:oposition 3.5. If supp(R,G) = G then $(-)(\ )_e$ is equivalent

› $1_{R\text{-Agr}}$.

:oof: Let β_M: $\overline{M}_e \longrightarrow M$ be given by $\beta_M(m\sigma) = r_\sigma\, m$. We show

$(-)(\ )_e \longrightarrow 1_{R\text{-Agr}}$ is a natural transformation and β_M is

ı equivalence in R-Agr.

Let $f: M \longrightarrow N$ be R-Agr map. Consider the diagram:

$$\begin{array}{ccc} \overline{M}_e & \xrightarrow{\ \beta_M\ } & M \\ {\scriptstyle \overline{f}_e}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{N}_e & \xrightarrow[\ \beta_N\ ]{} & N \end{array}$$

Let $m \in M_e$ and $m\sigma \in \overline{M}_e$. Then $\beta_N\ \overline{f}_e(m\sigma) = \beta_N\left(f_e(m)\sigma\right) = r_\sigma f_e(m) = r_\sigma f(m) = f(r_\sigma\ m) = f\ \beta_M(m\sigma)$, i.e., the diagram commutes and hence β is a natural transformation.

Since supp(R,G) = G, $M_\sigma = r_\sigma\ M_e$ by Remark 2.2 and hence every $x \in M_\sigma$ can be written as $r_\sigma\ m$ for some $m \in M_e$. Define $\gamma: M \longrightarrow \overline{M}_e$ by $\gamma(r_\sigma\ m) = m.\sigma$ where $m \in M_e$. Then clearly $\gamma \in$ R-Agr. Now, $\gamma\beta_M(m\sigma) = \gamma(r_\sigma\ m) = m\sigma$ and $\beta_M\ \gamma(r_\sigma\ m) = \beta_M(m\sigma) = r_\sigma\ m$. Therefore, β_M is an equivalence in R-Agr.

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