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Minimal KC-Spaces and Minimal LC-Spaces

Dr. Radhi I.M. Ali*

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□ ABSTRACT □

In this paper, we study KC-spaces; these are the spaces in which every compact subset is closed. Then we introduce the concept of minimal KC-spaces and we study the relation between minimal KC-spaces and minimal Hausdroff spaces. Finally, we introduce a new concept of minimal LC-spaces. Most of the theorems which are valid for minimal KC-spaces will also be valid for minimal LC-spaces.

***Assistant Prof., Department of Mathematics, Jerash Private University, Jerash, Jordan.**

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فضاءات KC الأصغرية وفضاءات LC الأصغرية

الدكتور راضي إبراهيم محمد علي*

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□ الملخّص □

يقال عن فضاء تبولوجي (X,τ) إنه فضاء KC إذا كانت كل مجموعة متراصة فيه مجموعة مغلقة ويقال عنه إنه فضاء LC إذا كان كل فضاء لندلوف جزئي منه يشكل مجموعة مغلقة فيه. ولقد قمنا في هذا البحث بدراسة فضاء KC و فضاء LC ثم قدمنا فضاءات KC الأصغرية و فضاءات LC الأصغرية ودرسنا العلاقة بين فضاءات هاوسدورف الأصغرية وفضاءات KC الأصغرية ثم بيّنا أن معظم النتائج المتحققة في حالة فضاءات KC تكون متحققة في حالة فضاءات LC.

* أستاذ مساعد في قسم الرياضيات- كلية الزراعة والعلوم- جامعة جرش الأهلية- جرش- الأردن.

1. Introduction:

Let R be a topological property, X be a nonempty and let R (X) denote the set of all topologies on X having the property R. R (X) is partially ordered by set inclusion. (X,τ) is minimal R (R-minimal) if τ is minimal in R (X). In [1] there is a good survey on minimal topologies and it stated there that every compact Hausdroff is minimal Hausdroff. In this paper we show that compact KC-space is minimal KC-space.

In this section we recall the basic facts concerning KC-spaces.

Definition 1.1 [2]:

Let (X,τ) be a topological space, we say that (X,τ) is a KC- space if every compact subset of is a closed in X.

Remarks and Examples 1.2:

1. Every Hausdroff space is a KC-space for example (**R**, τu) is a KC-space (where τu is the usual topology on **R**).
2. (**R**, τcc) is a KC-space but not a Hausdroff space (where τcc is the co-countable topology on **R**)
3. Every KC-space is a T1-space. So we have the following diagram
   T2-space → KC-spaces → T1-space
4. (**R**, τc) is a T1-space which is not a KC-space (where τc is the co-finite topology on **R**)
5. Let X be a finite set then X is a KC-space iff X is a T1-space.
6. Every continuous function from a compact space into a KC-space is a closed function.

Proposition 1.3:

Let X be a locally compact space then X is a KC-space iff X is a T2-space.

Proof:

Suppose that X is a KC-space, since X is a locally compact, then every neighborhood of $x \in X$ contains a compact neighborhood of x (some authors define locally compact in a different way).Hence the family of compact neighborhood of x in X will be a local base at $x \in X$, but X is a KC-space. Thus the family of closed neighborhood of $x \in X$ will be a local at $x \in X$. Therefore X is a regular, but X is a T1-space then X is a T3-space which implies that X is a T2-space. The converse is clear.□

2. Properties of KC-spaces:

In this section we state and prove several properties of KC-spaces.

Remark 2.1: The continuous image of KC-space is not necessarily a KC-space as shown by the following example: consider IR: (**R**, τu) → (**R**, τi)**,** where IR is the identity function on **R.** Now (**R**, τu) is a KC-space but (**R**, τi) is not KC-space, where τi is the indiscrete topology on **R**.

Proposition 2.2: If f: X → Y is a continuous injective function from X into a KC-space Y then X is KC-space, too.

Proof: Let W be any compact subset of X then f (W) is compact subset in Y. Since Y is KC-space, f (W) is closed subset of Y, see [3]. Therefore f^{-1} (f (W)) = W is closed subset of X, because f is continuous injective function. Thus X is KC-space.□

Proposition 2.3: The property of being KC-space is a topological property.
Proof: Let (X,τ_x) be a KC-space, f: $(X,\tau_x) \to (Y, \tau_Y)$ be a homeomorphism and let W be any compact subset of Y, then f^{-1} (W) is compact in X, but X is a KC-space, so f^{-1} (W) is closed in X then $f(f^{-1}(w))$ =W is closed in Y. Therefore Y is a KC-space.□

Proposition 2.4: The property of being KC-space is a hereditary property.
Proof: Let (X,τ_x) be a KC-space, (Y, τ_Y) be a subspace of X, and let $A \subseteq Y$ be any compact subset in Y, then A is compact in X, but X is a KC-space. Therefore A is closed in X. Thus $A \cap Y = A$ is closed in Y; hence Y is a KC-space.□

<u>3. Minimal KC-Spaces</u>:

In this section, we introduce the concept of minimal KC-space
First we recall the definition of minimal T2-space.

Definition 3.1 [4]: Let (X,τ) be a T2-space, we say that (X,τ) is a minimal T2-space (minimal Hausdroff space) iff $\tau^* \subset \tau$ implies (X,τ^*) is not a T2-space, (we will use MH to denote minimal Hausdroff space).

Definition 3.2: Let (X,τ) be a KC-space, we say that (X,τ) is a minimal KC-space iff $\tau^* \subset \tau$ implies (X,τ^*) is not a KC-space, (we will use MKC to denote minimal KC-space).

Theorem 3.3: Every compact KC-space is a MKC.
Proof: Let (X,τ) be a compact KC-space. Suppose X is not MKC i.e. there is a topology $\tau^* \subset \tau$ on X such that (X,τ^*) is KC-space. Let $I_X : (X,\tau) \to (X,\tau^*)$ be the identity function on X. I_X is a continuous, bijective and closed function, hence I_X is a homeomorphism implies that $\tau^* = \tau$ which is a contradiction so (X,τ) is MKC.□

Examples 3.4:
1. Consider I = [0 , 1] in ($\mathbf{R}$, τ_u). I is a T2-space so I is a KC-space. Since I is a compact space then, by theorem 3.3, I is a MKC.
2. Let X be a nonempty finite set then (X,τ_d) is MKC (where τ_d discrete topology on X).

Remark 3.5: The continuous image of MKC is not necessarily MKC, as shown by the following example, let X be a nonempty finite set and let $I_X : (X,\tau_d) \to (X,\tau_i)$ be the identity function on X. (X,τ_d) is MKC but (X,τ_i) is not MKC.

Proposition 3.6: The property of being MKC is a topological property.
Proof: Let (X,τ_x) be a MKC-space, f: $(X,\tau_x) \to (Y, \tau_Y)$ be a homeomorphism. Notice that (Y, τ_Y) is a KC-space and suppose that (Y, τ_Y) is not a MKC, then there exists a topology $\tau^*_Y \subset \tau_Y$ such that (Y, τ^*_Y) is a KC-space. Define $\tau_1 = \{f^{-1}(V): V \in \tau^*_Y\}$, τ_1 is a

topology on X and $\tau 1 \subset \tau x$ and (X, $\tau 1$) is a KC-space which is a contradiction with X is a MKC. Hence (Y, τY) is a MKC.□

Theorem 3.7: Let (X,τx) be a compact KC-space, and (Y, τY) be a subspace of X, then Y is compact iff Y is a closed set in X.
Proof: Suppose Y is compact, since X is KC-space then Y is closed. Conversely, suppose Y is a closed in X then Y is compact because X is compact.□

Corollary 3.8: Let (X,τx) be a compact KC-space, then every closed subspace of X is MKC.

Proposition 3.9: Every locally compact MKC is MH.
Proof: Let (X,τx) be a locally compact MKC-space, so X is a locally compact KC-space, hence X is a T2-space. Suppose X is not a MH-space, so there exists a topology τ^* on X, $\tau^* \subset \tau$ and (X,τ^*) is a T2-space implies that (X,τ^*) is a KC-space which is a contradiction. Therefore (X,τ) is MH.□

Proposition 3.10: Suppose X1 x X2 is a compact KC-space, then each of X1, X2 is a MKC-space.
Proof: Since X1 x X2 is a compact then each of X1, X2 is a compact, too. Let x*2 be a fixed element in X2, X1 x {x*2} is a subspace of X1 x X2, therefore X1 x {x*2} is a KC-space. But X1 is a homeomorphic to X1 x {x*2} implies that X1 is a KC-space. Thus X1 is a compact KC-space, by using theorem 3.3 X1 is MKC-space. Similarly we can show that X2 is a MKC-space. □

We can generalize above result to finite product X1 x X2 x…x Xn and to arbitrary products as follows:

Theorem 3.11: Let $\Im=\{X\alpha: \alpha \in \Omega\}$ be any family of topological spaces. If X= $\prod X\alpha$ is a compact KC-space, then each $X\alpha$ is a MKC-space for each $\alpha \in \Omega$.
Proof: Let $\alpha^* \in \Omega$ we will show that $X\alpha^*$ is MKC-space. Since X= $\prod X\alpha$ is a compact and the projection $P\alpha$: X→ $X\alpha$ is continuous function and the continuous image of the compact is compact, then $X\alpha$ is compact, in particular $X\alpha^*$is compact. Now, define Y=$\prod Y\alpha$ where

$$Y\alpha=\begin{cases} X\alpha^* & \text{if} \quad \alpha=\alpha^*, \\ x^*\alpha & \text{if} \quad \alpha\neq\alpha^*, \end{cases}$$

Where $x^*\alpha$ is a fixed point in $X\alpha$. $Y\alpha$ is a subspace of X so Y is a KC-space. Since $X\alpha^*$ is homeomorphic to Y then $X\alpha^*$ is a KC-space and by using theorem 3.3 $X\alpha^*$ is MKC-space. Because of α^* is arbitrary, therefore, $X\alpha$ is a MKC-space.□

4. Minimal LC-Spaces:

In this section, we introduce a new concept, namely minimal LC-space. First, we recall a few definitions and facts concerning LC-space.
Definition 4.1 [2]: Let (X,τ) be a topological space we say that X is LC-space if every Lindelöf subspace of X is closed in X.

Remarks and Examples 4.2:

1. Every LC-space is a KC-space, hence every LC-space is a T1-space, i.e.

LC
↓
T2 → KC → T1

2. Every T2-p-space is LC-space (where p-space is the space in which countable intersection of open sets is open set.)

Proposition 4.3: Every locally compact LC-space is a T2-space.
Proof: Let X be a locally compact LC-space, then X is a locally compact KC-space. Hence X is a T2-space. □

Remark 4.4: The continuous image of LC-space is not necessarily LC-space as shown by the following example. Consider IR: (**R**, τ_d) → (**R**, τ_i) is the identity function on **R**, notice that (**R**, τ_d) is a LC-space but (**R**, τ_i) is not LC-space.

Proposition 4.5: If f: X → Y is a continuous injective function from X into a LC-space Y then X is LC-space, too.
Proof: Let W be any Lindelöf subset of X then f (W) is Lindelöf subset in Y. Since Y is LC-space, f (W) is closed subset of Y, see [4]. Therefore $f^{-1}(f(W)) = W$ is closed subset of X, because f is continuous injective function. Thus X is LC-space. □

Proposition 4.6: The property of being LC-space is a topological property.
Proof: Let (X,τ_x) be a LC-space, f: $(X,\tau_x) \to (Y, \tau_Y)$ be a homeomorphism and let $A \subseteq Y$ be a Lindelöf subset. Since $f^{-1}(A)$ is a Lindelöf subset of X and X is LC-space, then $f^{-1}(A)$ is closed in X. Thus, $f(f^{-1}(A)) = A$ is closed in Y, therefore, Y is LC-space. □

Proposition 4.7: The property of being LC-space is a hereditary property.
Proof: Let (X,τ_x) be a LC-space, (Y, τ_Y) be a subspace of X and let $A \subseteq Y$ be a Lindelöf subset of Y. Therefore, A is a Lindelöf subset of X, implies that A is closed in X because X is a LC-space. But $A = A \cap Y$ is closed in Y i.e. Y is a LC-space. □

Now, we introduce the definition of minimal LC-space.

Definition 4.8: Let (X,τ) be a LC-space we say that X is a minimal LC-space (MLC) iff $\tau^* \subset \tau$ implies (X,τ^*) is not LC-space.

Theorem 4.9: Every Lindelöf LC-space is a MLC.
Proof: Let (X,τ) be a Lindelöf LC-space and suppose (X,τ) is not MLC, then there is a topology τ^* on X such that $\tau^* \subset \tau$ and (X,τ^*) is LC-space. Let $I_X : (X,\tau) \to (X,\tau^*)$ be the identity function on X. I_X is a continuous, bijective and closed function then I_X is a homeomorphism which implies that $\tau^* \cong \tau$, but this is a contradiction, so (X,τ) is MLC. □

Example 4.10: Let X be a countable set, then (X, τ_d) is a MLC.

Remark 4.11: The continuous image of MLC is not necessarily MLC, as shown by the following example: Let X a countable set and let Ix : $(X,\tau d) \rightarrow (X,\tau i)$ be the identity function on X. $(X,\tau d)$ is MLC but $(X,\tau i)$ is not MLC.

Theorem 4.12: The property of being MLC-space is a topological property.
Proof: Let $(X,\tau x)$ be a MLC-space, f: $(X,\tau x) \rightarrow (Y, \tau Y)$ be a homeomorphism**.** Notice that $(Y, \tau Y)$ is LC-space and suppose it is not MLC then there exists a topology τ^*Y on Y such that $\tau^*Y \subset \tau Y$ and $(Y, \tau Y)$ is LC-space. Define $\tau^*x = \{f(A): A \in \tau^*Y\}$, τ^*x is a topology on X, $\tau^*x \subset \tau x$ and (X,τ^*x) is a LC-space which is a contradiction. Hence $(Y,\tau Y)$ is a MLC. □

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Theorem 4.13: Let $(X,\tau x)$ be a Lindelöf LC-space and let $(Y, \tau Y)$ be a subspace of X then $(Y, \tau Y)$ is a Lindelöf iff Y is closed in X.
Proof: Suppose $(Y, \tau Y)$ is a Lindelöf space, but X is LC-space therefore Y is closed in X. Conversely, suppose $(Y, \tau Y)$ is closed in X, but X is Lindelöf space, then Y is also Lindelöf space. □

Corollary 4.14: Let $(X,\tau x)$ be a Lindelöf LC-space and let $(Y, \tau Y)$ be a closed subspace of X then $(Y, \tau Y)$ is a MLC.

Theorem 4.15: Let X1 x X2 be a Lindelöf LC-space, then each of X1, X2 is a MLC-space.
Proof: Let x*2 be any fixed element in X2.Then X1 x {x*2} is a subspace of X1 x X2, by proposition 4.7, X1 x {x*2} is a LC-space. But X1 x {x*2} is homeomorphic to X1, by proposition 4.5, X1 is LC-space, too. Therefore, by proposition 4.9, X1 is MLC. □

The above theorem can be generalized to finite product and arbitrary product as follows.

Theorem 4.16: Let $\Im=\{X\alpha: \alpha \in \Omega\}$ be any family of topological spaces. If $X=\prod X\alpha$ is a Lindelöf LC-space then each $X\alpha$ is MLC-space for each $\alpha \in \Omega$.
Proof: Let $\alpha^* \in \Omega$ we will show that $X\alpha^*$ is MLC-space. Since $X=\prod X\alpha$ is a Lindelöf and the projection $P\alpha: X\rightarrow X\alpha$ is continuous function and the continuous image of the Lindelöf is Lindelöf, then $X\alpha$ is Lindelöf, in particular $X\alpha^*$is Lindelöf. Now, define $Y=\prod Y\alpha$ where

$$Y\alpha=\begin{cases} X\alpha^* & \text{if } \alpha=\alpha^*, \\ x^*\alpha & \text{if } \alpha\neq\alpha^*, \end{cases}$$

Where $x^*\alpha$ is a fixed point in $X\alpha$. $Y\alpha$ is a subspace of X so Y is a LC-space. Since $X\alpha^*$ is homeomorphic to Y then $X\alpha^*$ is a LC-space and by using theorem 4.9 $X\alpha^*$ is MLC-space. Because of α^* is an arbitrary, therefore, $X\alpha$ is a MLC-space.□

5. Open problems:

In this section we are going to establish some open problems arise, concerning the minimal KC-space and minimal LC-space,

1. Under what conditions the continuous image of minimal KC-space is minimal KC-space, too.
2. Under what conditions the continuous inverse image of minimal KC-space is minimal KC-space, too.
3. Under what conditions the continuous image of minimal LC-space is minimal LC-space, too.
4. Under what conditions the continuous inverse image of minimal LC-space is minimal LC-space, too.

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