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A Class of Three-Point Spline Collocation Methods for Solving Delay-Differential Equations

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□ ABSTRACT □

In this paper a study of the existence, uniqueness, stability and convergence of a class of C^2-spline collocation methods for solving delay differential equations (DDEs) is introduced. The presented methods are based on C^2-Spline with three collocation points $x_{i-1+c_j} = x_{i-1} + c_j h,\ j = 1(1)3$, $c_1, c_2 \in (0,1)$, $c_1 \neq c_2$ and $c_3 = 1$ in each subinterval $\mathrm{I}_i = [x_{i-1}, x_i],\ i = 1(1)\mathrm{N}$. It turns out that the proposed methods for DDEs are stable iff $c_1 + c_2 \geq 1$, and they possess convergence rate of order 6 if $58 - 57(c_1 + c_2) + 55c_1c_2 = 0$, in the remaining cases the order is 5. Moreover, the methods are P-stable for $0.8028 \leq c_1 < c_2 < 1$. Numerical results illustrating the behavior of the methods when faced with some difficult problems are presented and the numerical results are compared to those obtained by other methods.

Key words. *Delay differential equations; spline collocation methods; stability analysis, convergence.*

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صف من الطرائق الشرائحية بثلاث نقاط تجميعية لحل معادلات تفاضلية متأخرة

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☐ الملخّص ☐

نقدم في هذا البحث صفاً من الطرائق الشرائحية التجميعية لإيجاد الحل العددي للمعادلات التفاضلية المتأخرة. تعتمد الطرائق المذكورة على إنشاء تقريبات شرائحية في الفضاء C^2 باستخدام ثلاث نقاط تجميعية $x_{i-1+c_j} = x_{i-1} + c_j h,\ j = 1(1)3$ في كل مجال جزئي $I_i = [x_{i-1}, x_i],\ i = 1(1)N$ حيث $c_1, c_2 \in (0,1)$ و $c_1 \neq c_2$ ، $c_3 = 1$. تم إثبات وجود حل تقريبي شرائحي وحيد لمثل هذه المعادلات، وجرت دراسة استقرار وتقارب ومعدل التقارب لهذه الطرائق. تبين الدراسة أن الطرائق لأجل المعادلات المذكورة تكون مستقرة إذا كان $c_1 + c_2 \geq 1$ ، علاوة على ذلك ، الطرائق تكون متقاربة، وهذا التقارب من المرتبة السادسة لأجل بارامترات تحقق المعادلة $58 - 57(c_1 + c_2) + 55c_1c_2 = 0$ ، وفي حالات أخرى يكون التقارب من المرتبة الخامسة. بالإضافة إلى ذلك، يظهر تحليل الاستقرار أن الطرائق تكون في حالة P–استقرار لأجل $0.8028 \leq c_1 < c_2 < 1$. كما تم اختبار الطرائق المقدمة بحل بعض المسائل ذات السلوك القاسي ولأجل دالة بدء إما أن تكون غير ملساء أو تملك تذبذبات عالية، حيث تشير النَتائِج العددية إلى فعالية وكفاءة طرائقنا مقارنة مع بعض الطرائقِ الأخرى.

الكلمات المفتاحية: *معادلات تفاضلية متأخرة، طرائق شرائحية تجميعية، تحليل الاستقرارية، التقارب.*

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1. Introduction

Delay differential equations (DDEs) arise in many areas of mathematical modeling: for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for example, the body's reaction to CO_2, etc. in circulating blood) as well as chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft (with respectively large and short lags), and more general control problems.

1.1 Contributions of the work

The main contribution of this work is the development and analysis of spline collocation methods with three collocation conditions of the numerical solution of DDE:

$$y'(x) = f(x, y(x), y(\alpha(x))), \quad x \in [a,b]$$

Where $f \in C^5([a,b]\times R\times R)$ is Lipschitiz continuous with respect *y*. The function $\alpha(x) \le x, x \in [a,b]$ is usually called the delay function. For $\tilde{a} = \inf[\alpha(x)]$, we assume that the initial condition is given by $y(x) = g(x)$, $x \in [\tilde{a}, a]$ for a given function $g(x)$.

1.2 A review of previous work

The stability of numerical methods for DDEs has previously been considered in [8, 10, 18, 19] based on the linear DDE

$$y'(x) = \lambda\, y(x) + q\; y(x-\tau), \quad x > 0, \tag{1.1a}$$

$$y(x) = g(x), \qquad x \le 0 \tag{1.1b}$$

Where $\lambda, q \in C$, $\tau > 0$, and $g(x)$ is an initial function. It is known that if $g(x)$ is continuous and if $|q| < -\mathrm{Re}(\lambda)$, then the solution $y(x)$ to (1.1a)-(1.1b) tends to zero as $x \to \infty$.

In't Hout [12] has considered adaptation of the class of Runge-Kutta methods for DDEs. The numerical stability of linear multistep formulas has been studied in [8,9,18]. Results on the P-stability and GP-stability of some numerical methods have been given in [8, 19]. Using continuous Runge-Kutta methods for numerical solution of retarded and neutral DDEs by Hayasshi [11]. Engelborghs et al. [5] have introduced collocation methods for the computation of periodic solutions of DDEs.

A collocation procedure with polynomial spline functions of degree $m \ge 3$ and continuity class C^{m-2} is considered for numerical solution of a second initial value problem for neutral DDEs by Akca et al. [1].

El-Hawary & Mahmoud [2, 3, 4] presented C^3-Spline collocation methods for solving ordinary and algebraic differential equations including stiff differential equations. They showed that the method is successfully applied in [2] for solving systems of stiff equations and in [3] the method is accurate for solving dynamical systems, also in [4] the method is effectively applied for solving higher index differential-algebraic equations.

1.3 An outline of the paper

The paper is organized as follows. In **Section 2** the precise description of spline collocation methods is provided. In addition, it contains an investigation of the existence and uniqueness of the proposed methods when applied to DDEs. Sufficient conditions for the convergence of the methods are given in **Section 3**. A detailed study for stability is presented in **Section 4**. Finally, we conclude with numerical test examples and conclusion in **Section 5** and **6**.

2. Description of spline collocation methods for DDEs.

Consider the following initial value problem for DDEs

$$y'(x) = f\{x, y(x), y[\alpha(x)]\}, \quad x \in [a,b] \tag{2.1a}$$

$$y(x) = g(x) \quad \text{for} \quad \tilde{a} \le x \le a \tag{2.1b}$$

where $f \in C^5([a,b]\times R \times R)$ is Lipschitiz continuous with respect y, $\tilde{a} = \inf[\alpha(x)]$.

The spline methods use three-collocation points $x_{i-1+c_j} = x_{i-1} + c_j h,\ j = 1(1)3$ in each subinterval $[x_{i-1}, x_i]$, $i = 1(1)\mathrm{N}$, with

$$0 < c_1 < c_2 < 1 \tag{2.2}$$

and $h=(b-a)/N$ is the constant stepsize, where $c_3 = 1$, $x_0 = a$, $x_\mathrm{N} = b$.

Denote by $x_i = a + i\,h,\ i = 0(1)\mathrm{N}$, the grid points of the uniform partition of $[a, b]$ into subintervals $\mathrm{I}_i = [x_{i-1}, x_i],\ i = 1(1)\mathrm{N}$.

A quintic C^2-spline functions $S(x)$ can be represented on each I_i by [17]

$$\begin{aligned} S(x) = c'^3 &[(6c^2 + 3c + 1)S_{i-1}^{(0)} + (3c^2 + c)S_{i-1}^{(1)} + (\tfrac{1}{2}c^2)S_{i-1}^{(2)}] \\ &+ c^3[(6c'^2 + 3c' + 1)S_i^{(0)} - (3c'^2 + c')S_i^{(1)} + (\tfrac{1}{2}c'^2)S_i^{(2)}] \end{aligned} \tag{2.3}$$

where $c = \dfrac{(x - x_{i-1})}{h} \in [0,1]$, $c' = 1 - c$ and

$$S_i^{(0)} = S(x_i),\ S_i^{(1)} = hS'(x_i),\ S_i^{(2)} = h^2 S''(x_i),\quad i = 0(1)\mathrm{N} \tag{2.4}$$

Differentiating (2.3), we get

$$\begin{aligned} h\,S'(x) = c'^2 &[-30\,S_{i-1}^{(0)} + (1 + 2c - 15c^2)S_{i-1}^{(1)} + (c - \tfrac{5}{2}c^2)S_{i-1}^{(2)}] \\ &- c^2[-30\,c'^2\,S_i^{(0)} - (1 + 2c' - 15c'^2)S_i^{(1)} + (c' - \tfrac{5}{2}c'^2)S_i^{(2)}] \end{aligned} \tag{2.5}$$

We formally apply these methods to (2.1a), for $S(x)$ to be satisfied by the three collocation conditions:

$$S'(x_{i-1+c_j}) = f\{x_{i-1+c_j}, S(x_{i-1+c_j}), S[\alpha(x_{i-1+c_j})]\} \quad, j = 1(1)3 \tag{2.6}$$

in each subinterval $[x_{i-1}, x_i]$.

More precisely, denoting $f_{i-1+\phi} \equiv f\{x_{i-1+\phi}, S(x_{i-1+\phi}), S[\alpha(x_{i-1+\phi})]\},\ 0 \le \phi \le 1$, and $c'_j = 1 - c_j$, we can write (2.6) as follows:

$$\begin{aligned} &c_j^2[30c_j'^2\,S_i^{(0)} + (1 + 2c'_j - 15c_j'^2)S_i^{(1)} - (c'_j - \tfrac{5}{2}3c_j'^2)S_i^{(2)}] \\ &= c_j'^2[30c_j^2\,S_{i-1}^{(0)} - (1 + 2c_j - 15c_j^2)S_{i-1}^{(1)} - (c_j - \tfrac{5}{2}c_j^2)S_{i-1}^{(2)}] \\ &\quad + hf_{i-1+c_j} \quad, j = 1(1)3, \end{aligned} \tag{2.7}$$

Substituting $S_i^{(1)} = hf_i$, $S_{i-1}^{(1)} = hf_{i-1}$ into (2.7) and dividing by $30c_j^2c_j'^2$, we get the equivalent recurrence formulae:

$$\begin{cases} \underline{S}_i = A\underline{S}_{i-1} + hB\underline{f}_i, \\ S_i^{(1)} = hf_i \quad, i = 1(1)N \end{cases} \tag{2.8}$$

where $A=\tilde{A}^{-1}D=\begin{bmatrix}1 & \dfrac{5(c_1c_2+c_1'c_2')-2}{60c_1c_2}\\ 0 & -\dfrac{c_1'c_2'}{c_1c_2}\end{bmatrix}$, (2.9)

$$B=\begin{bmatrix}1 & \frac{1}{12}-\frac{1}{30c_1'}\\ 1 & \frac{1}{12}-\frac{1}{30c_2'}\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{15c_1^2-2c_1-1}{30c_1^2} & \dfrac{1}{30c_1^2c_1'^2} & 0 & \dfrac{15c_1'^2-2c_1'-1}{30c_1'^2}\\ \dfrac{15c_2^2-2c_2-1}{30c_2^2} & 0 & \dfrac{1}{30c_1^2c_1'^2} & \dfrac{15c_2'^2-2c_2'-1}{30c_2'^2}\end{bmatrix}, \quad (2.10)$$

$$\tilde{A}^{-1}=\begin{bmatrix}1 & \frac{1}{12}-\frac{1}{30c_1'}\\ 1 & \frac{1}{12}-\frac{1}{30c_2'}\end{bmatrix}^{-1}, \qquad D=\begin{bmatrix}1 & \frac{1}{12}-\frac{1}{30c_1}\\ 1 & \frac{1}{12}-\frac{1}{30c_2}\end{bmatrix},$$

$$\underline{S}_i=(S_i^{(0)},S_i^{(2)})^T,\ \underline{f}_i=(f_{i-1},f_{i-1+c_1},f_{i-1+c_2},f_i)^T.$$

It is easy to observe that initial condition (2.1b) becomes

$$S[\alpha(x_{i-1+c_j})]=g[\alpha(x_{i-1+c_j})],\quad \alpha(x_{i-1+c_j})\le a, \quad (2.11a)$$

and if $\alpha(x_{i-1+c_j})\in[x_{k-1},x_k]$, $k\le i$, then $S[\alpha(x_{i-1+c_j})]$ can be calculated from (2.3):

$$\begin{aligned}S[\alpha(x_{i-1+c_j})]&=\zeta'^3_{k-1+c_j}[(6\zeta^2_{k-1+c_j}+3\zeta_{k-1+c_j}+1)S^{(0)}_{k-1}\\&+(3\zeta^2_{k-1+c_j}+\zeta_{k-1+c_j})S^{(1)}_{k-1}+(\tfrac{1}{2}\zeta^2_{k-1+c_j})S^{(2)}_{k-1}]\\&+\zeta^3_{k-1+c_j}[(6\zeta'^2_{k-1+c_j}+3\zeta'_{k-1+c_j}+1)S^{(0)}_{k}\\&-(-3\zeta'^2_{k-1+c_j}+\zeta'_{k-1+c_j})S^{(1)}_{k}+(\tfrac{1}{2}\zeta'^2_{k-1+c_j})S^{(2)}_{k}],\quad j=1(1)3,\end{aligned} \quad (2.11b)$$

where $\zeta_{k-1+c_j}=\dfrac{\alpha(x_{i-1+c_j})-x_{k-1}}{h}$ and $\zeta'_{k-1+c_j}=1-\zeta_{k-1+c_j}$.

Since $\alpha(x_{i-1+c_j})-x_{k-1}\le hc_j$, let $\alpha(x_{i-1+c_j})-x_{k-1}=r\,h\,c_j$, (where $0<r\le 1$), then $\zeta_{i-1+c_j}=\dfrac{r\,h\,c_j}{h}=r\,c_j$ and $\zeta'_{k-1+c_j}=1-r\,c_j$.

If $0<c_1<c_2<1$, then $\tilde{A}^{-1}$ is nonsingular because $|\tilde{A}|=\dfrac{(c_1-c_2)}{30(c_1-1)(c_2-1)}\ne 0$

Theorem 1: If $f\in C^5([0,b]\times\Re\times\Re)$ satisfies Lipschitz condition, and if

$h<1/RL$ (2.12)

then there exists a unique spline approximation solution of (2.1) given by (2.8) for all c_1, c_2 satisfying (2.2).

Proof. It is sufficient to prove that $\underline{S}_i=(S_i^{(0)},S_i^{(2)})^T$ can be uniquely determined for arbitrary given $\underline{S}_{i-1}$.

Let $\underline{S}_{i,1},\underline{S}_{i,2}\in R^2$, then using $\|\cdot\|_1$ from (2.8), we have

$$\underline{S}_{i,1} = A\underline{S}_{i-1} + hB\underline{f}_{i,1} \text{ and } \underline{S}_{i,2} = A\underline{S}_{i-1} + hB\underline{f}_{i,2}$$

Thus $\underline{S}_{i,1}$ and $\underline{S}_{i,2}$ can be written in the form

$$\underline{S}_{i,1} = \underline{Q}_{i,1}(s_{i,1}^{(0)}, s_{i,1}^{(2)}, h) \text{ and } \underline{S}_{i,2} = \underline{Q}_{i,2}(s_{i,2}^{(0)}, s_{i,2}^{(2)}, h)$$

Applying $\|\,.\,\|_1$, Lipschitz condition and using Mathematica, we get

$$\|\underline{Q}_{i,1} - \underline{Q}_{i,2}\| = \|(A\underline{S}_{i-1} + hB\underline{f}_{i,1}) - (A\underline{S}_{i-1} + hB\underline{f}_{i,2})\|$$

$$\le \{\frac{1}{10}|[325r^3 - 630r^4 + (252 + 57(c_1 + c_2) - 55c_1c_2)r^5 - 58 + 57(c_1 + c_2) - 55c_1c_2]|\,hL_0\,|s_{i,1}^{(0)} - s_{i,2}^{(0)}| +$$

$$+\frac{1}{120}|[195r^3 - 504r^4 + (252 + 57(c_1 + c_2) - 55c_1c_2)r^5 - 58 + 57(c_1 + c_2) - 55c_1c_2]|\,hL_2\,|s_{i,1}^{(2)} - s_{i,2}^{(2)}|\}$$

$$< RhL\{|s_{i,1}^{(0)} - s_{i,2}^{(0)}| + |s_{i,1}^{(2)} - s_{i,2}^{(2)}|\}$$

where

$$L = \max(L_0, L_2),\ R = \max(R_1, R_2),$$

$$R_1 = \tfrac{1}{10}|[325r^3 - 630r^4 + (252 + 57(c_1 + c_2) - 55c_1c_2)r^5 - 58 + 57(c_1 + c_2) - 55c_1c_2]|,$$

$$R_2 = \tfrac{1}{120}|[195r^3 - 504r^4 + (252 + 57(c_1 + c_2) - 55c_1c_2)r^5 - 58 + 57(c_1 + c_2) - 55c_1c_2]|$$

Thus, the function $\underline{Q}_i$ defines a contraction mapping if $RhL < 1$ which satisfies (2.12). Hence there exists a unique $\underline{S}_i$ that satisfies $\underline{S}_i = \underline{Q}_i(s_i^{(0)}, s_i^{(2)}, h)$

which may be found by iteration,

$\underline{S}_i^{p+1} = \underline{Q}_i(\underline{S}_i^p, h)$, p=0,1,2,…

The proof of the theorem1 is now complete.

3. Error Analysis and order of convergence

In this section we consider the convergence of the methods (2.4), (2.6) with initial condition (2.11a). To find a numerical approximation $S(x)$ to the exact solution y, we define $S(x)=g(x)$ for $x \le a$. The spline methods produce function values $S(x_i)$ as approximation to $y(x_i)$. The unknown value $y(\alpha(x))$ may be replaced by $S(\alpha(x))$.

Theorem 2: The methods (2.4), (2.6), (2.11a) are stable iff.

$$c_1 + c_2 \ge 1 \qquad (3.1)$$

Proof. According to the definition of stability (definition 8.8 in [6]) we have to check the uniform boundedness of $\{A^n\}$ where A is the matrix (2.9). Since $|a_{ij}| \le 1$,

(i,j=1,2), n>0, then $\| A^n \| \le k$, $k = \max_{1\le i\le 2} \sum_{j=1}^{2} | a_{ij}^n |$. On the other hand, A has two different eigenvalues $\lambda_1 = 1$ and $\lambda_1 = -\frac{c_1' c_2'}{c_1 c_2}$. Thus, the methods are stable iff $|\lambda_i| \le 1$, i=1,2, implies that $c_1' c_2' \le c_1 c_2$ which is equivalent to (3.1).

<u>Theorem 3:</u> Let $f \in C^6([a,b]\times R\times R)$**,** then the methods (2.4),(2.6),(2.11) are consistent and are of order six iff**.**

$$58 - 57c_1 - 57c_2 + 55c_1 c_2 = 0 \qquad (3.2)$$

Moreover, if $f \in C^5([a,b]\times R\times R)$, then in the remaining cases the methods are consistent and are of order five.

Proof. Let $\alpha(x_{i-1+c_j}) \in [x_{k-1}, x_k]$, then we have the discretization error

$$\underline{d}_i = \begin{bmatrix} y(x_i) \\ h^2 y''(x_i) \end{bmatrix} - A \begin{bmatrix} y(x_{i-1}) \\ h^2 y''(x_{i-1}) \end{bmatrix} - hB \begin{bmatrix} f[x_{i-1}\ ,\ p_i(x_{i-1})\ ,\ p_k(x_{k-1})] \\ f[x_{i-1+c_1}, p_i(x_{i-1+c_1}), p_k(x_{k-1+c_1})] \\ f[x_{i-1+c_2}, p_i(x_{i-1+c_2}), p_k(x_{k-1+c_2})] \\ f[x_i\ ,\ p_i(x_i)\ ,\ p_k(x_k)] \end{bmatrix},$$

$$i = 1(1)\mathrm{N}, \quad k \le i,$$

where

$$p_i(x) = c'^3 [(6c^2 + 3c + 1) y(x_{i-1}) + (3c^2 + c) y'(x_{i-1}) h + \tfrac{1}{2} c^2 y''(x_{i-1}) h^2]$$
$$+ c^3 [(6c'^2 + 3c' + 1) y(x_i) - (3c'^2 + c') y'(x_i) h + \tfrac{1}{2} c'^2 y''(x_i) h^2] ,$$

$$P_k(x) \equiv S[\alpha(x)] = \zeta'^3 [(6\zeta^2 + 3\zeta + 1) S_{k-1}^{(0)} + (3\zeta^2 + \zeta) S_{k-1}^{(1)} + (\tfrac{1}{2}\zeta^2) S_{k-1}^{(2)}]$$
$$+ \zeta^3 [(6\zeta'^2 + 3\zeta' + 1) S_k^{(0)} - (-3\zeta'^2 + \zeta') S_k^{(1)} + (\tfrac{1}{2}\zeta'^2) S_k^{(2)}], \quad j = 1(1)3,$$

$$\zeta = \frac{\alpha(x) - x_{k-1}}{h} \text{ and } \zeta' = 1 - \zeta .$$

is the quintic Hermite interpolation polynomial which interpolates y, y', y'' at $x = x_{i-1}$ and $x = x_i$, i=1(1)N.

Since

$|p_i(x) - y(x)| \le Lh^6, x \in [x_{i-1}, x_i]$, i=1(1)N

It follows that

$$\underline{d}_i = \underline{\tilde{d}}_i + o(h^7),\ i=1(1)\mathrm{N},$$

where

$$\underline{\tilde{d}}_i = \begin{bmatrix} y(x_i) \\ h^2 y''(x_i) \end{bmatrix} - A \begin{bmatrix} y(x_{i-1}) \\ h^2 y''(x_{i-1}) \end{bmatrix} - hB \begin{bmatrix} y'(x_{i-1}) \\ y'(x_{i-1+c_1}) \\ y'(x_{i-1+c_2}) \\ y'(x_i) \end{bmatrix}$$

Now using Taylor's expansion

$$y(x) = q_5(x) + \frac{h^6}{6!} y^{(6)}(x_{i-1}) c^6 + O(h^7), x \in [x_{i-1}, x_i],\ y \in C^7[0,b],$$

where $q_5(x)=\sum_{k=0}^{5}\frac{h^k}{k!}y^{(k)}(x_{i-1})c^k$

and observing that the methods are exact for polynomials of degree ≤ 5 (that means for

$y\equiv q_5$ we have $\tilde{\underline{d}}=\underline{d}_i=0$) we deduce, according to lemma 8.11(cf.(8.16)) in [6], that the methods are thus consistent and are of order at least five for all $c_1+c_2\ge 1$. Moreover, to get the statement in the exceptional cases in Table(1) we have to check

$$E\tilde{\underline{d}}_6(x)\equiv\begin{bmatrix}1&1\\0&0\end{bmatrix}\left\{\begin{bmatrix}\frac{1}{6!}\\ \\ \frac{1}{4!}\end{bmatrix}-A\begin{bmatrix}0\\ \\ \\ 0\end{bmatrix}-\frac{1}{5!}B\begin{bmatrix}0\\c_1^5\\c_2^5\\1\end{bmatrix}\right\}f^{(6)}(x)=0$$

Using the form (2.10) of B, we get

$$E\tilde{\underline{d}}_6=\begin{bmatrix}\frac{1}{60}(58-57c_1-57c_2+55c_1c_2)\\ \\ 0\end{bmatrix}$$

But if we let $58-57c_1-57c_2+55c_1c_2=0$, we get values listed in Table(1). This completes the proof.

Table(1): Some cases that make the methods of order six

c_1=0.51	c_2=0.999309	c_1=0.70	c_2=0.978378
c_1=0.55	c_2=0.996262	c_1=0.75	c_2=0.968254
c_1=0.60	c_2=0.991667	c_1=0.80	c_2=0.953846
c_1=0.65	c_2=0.985882	c_1=0.89	c_2=0.903106

Theorem 4: Let $f\in C^6([a,b]\times R\times R)$ be Lipschitz continuous. Then the spline approximation $S(x)$ given by (2.4), (2.6),(2.11) converges to the solution $y(x)$ of (2.1) as $h\to 0$ whenever (3.2) is fulfilled and

$$\lim_{h\to 0}h^{-j}S_0^{(j)}=y^{(j)}(x_0),\quad j=0(1)3$$

Furthermore, the convergence order is six, i.e., we have

$$|\,y^{(k)}(x_i)-\frac{1}{h^k}S_i^{(k)}\,|\le L_k h^6,\qquad k=0,1, \tag{3.3a}$$

$$|\,y^{(2)}(x_i)-\frac{1}{h^2}S_i^{(2)}\,|\le L_2 h^4,\quad i=1(1)\mathrm{N} \tag{3.3b}$$

whenever the initial values (2.4) satisfy (3.3a)and (3.3b) (with i=0). In addition for $x\ne x_i$, the following global error estimates hold true:

$$\left|y^{(k)}(x)-S_i^{(k)}(x)\right|\le L_k h^{6-k},\ \ k=0(1)5,\ \ x\in[a,b]. \tag{3.4}$$

Proof . Using Lipschitz condition, we have

$$|y'(x_i)-S'(x_i)|=|f\{x_i,y(x_i),y[\alpha(x_i)]\}-f\{x_i,S(x_i),S[\alpha(x_i)]\}|$$
$$\le L\{|y(x_i)-S(x_i)|+|y[\alpha(x_i)]-S[\alpha(x_i)]|\}$$
$$\le L\{L_0h^6+L_0h^6\}=L_1h^6\,,\qquad \alpha(x_i)\in[a,b]$$

where L_1=2L L_0.

Remark 1: Relations (3.4) follow from (3.3a) and (3.3b) in a straightforward manner using the quintic Hermite spline interpolate of $y(x)$.

4. Stability analysis

Let us consider the following linear delay differential equation

$$y'(x)=\lambda\, y(x)+q\; y(x-\tau) \qquad (4.1)$$

as stability test equation, where $\lambda, q\in C$ arbitrary, the delay τ is positive constant.

Definition1 A numerical method, applied to (4.1) is said to be P-stable if under the condition $\mathrm{Re}(\lambda)<-|q|$, the numerical solution $s(x_i)\to 0$ as $x_i\to\infty$ for all h satisfying $\tilde{m}h=\tau\,,\tilde{m}\in \mathrm{N}$. A region of P-stability is the set all points $(h\lambda, hq)$ for which the method is P-stable.

Applying the methods to (4.1)

$$S'(x_{i-1+c_j})=\lambda\, S(x_{i-1+c_j})+q\,S(x_{i-m-1+c_j}),\ j=1(1)3,\ i=1(1)\mathrm{N},\ m<i, \qquad (4.2)$$

where $\tau=mh$, $S(x_{i-m-1+c_j})=S(x_{i-1+c_j}-mh)$ and $x_{i-m-1+c_j}\in[x_{i-m-1},x_{i-m}]$,

we get from (2.3)-(2.5),(2.11):

$$\begin{aligned}
&\left[c_j^2(30c_j'^2)-z\,c_j^3(6c_j'^2+3c_j'+1)\right]S_i^{(0)}\\
&+\left[c_j^2(1+2c_j'-15c_j'^2)+z\,c_j^3(3c_j'^2+c_j')\right]S_i^{(1)}\\
&+\left[c_j^2(\tfrac{5}{2}c_j'^2-c_j')-z\,c_j^3(\tfrac{1}{2}c_j'^2)\right]S_i^{(2)}\\
&+\left[c_j'^2(-30c_j^2)-z\,c_j'^3(6c_j^2+3c_j+1)\right]S_{i-1}^{(0)}\\
&+\left[c_j'^2(1+2c_j-15c_j^2)-z\,c_j'^3(3c_j^2+c_j)\right]S_{i-1}^{(1)}\\
&+\left[c_j'^2(c_j-\tfrac{5}{2}c_j^2)-z\,c_j'^3(\tfrac{1}{2}c_j^2)\right]S_{i-1}^{(2)}=\\
&v\,c_j^3\left[(6c_j'^2+3c_j'+1)S_{i-m}^{(0)}-(3c_j'^2+c_j')S_{i-m}^{(1)}+(\tfrac{1}{2}c_j'^2)S_{i-m}^{(2)}\right]\\
&+v\,c_j'^3\left[(6c_j^2+3c_j+1)S_{i-m-1}^{(0)}+(3c_j^2+c_j)S_{i-m-1}^{(1)}+(\tfrac{1}{2}c_j^2)S_{i-m-1}^{(2)}\right] \qquad (4.3a)
\end{aligned}$$

$$j=1,2$$

$$S_i^{(1)}=zS_i^{(0)}+vS_{i-m}^{(0)}\,, \qquad (4.3b)$$

$$S_{i-1}^{(1)}=zS_{i-1}^{(0)}+vS_{i-m-1}^{(0)}\,. \qquad (4.3c)$$

where $z=\lambda h,\ v=qh$.

Or in matrix notation, (4.3a)-(4.3c) will be

$$A_1\bar{S}_i+A_2\bar{S}_{i-1}=B_1\bar{S}_{i-m}+B_2\bar{S}_{i-m-1} \qquad (4.4)$$

where $A_1=(a_{k,j}^1)$, $A_2=(a_{k,j}^2)$, $B_1=(b_{k,j}^1)$ and $B_2=(b_{k,j}^2)$ are defined by

$$a_{k,1}^{1}=30c_k^2c_k'^2-z\,c_k^3(6c_k'^2+3c_k'+1),$$
$$a_{k,2}^{1}=c_k^2(1+2c_k'-15c_k'^2)+z\,c_k^3(3c_k'^2+c_k'),$$
$$a_{k,3}^{1}=c_k^2(\tfrac{5}{2}c_k'^2-c_k')-\tfrac{1}{2}z\,c_k^3c_k'^2\ ,$$
$$a_{k,1}^{2}=-30c_k^2c_k'^2-z\,c_k'^3(6c_k^2+3c_k+1),$$
$$a_{k,2}^{2}=c_k'^2(1+2c_k-15c_k^2)-z\,c_k'^3(3c_k^2+c_k),$$
$$a_{k,3}^{2}=c_k'^2(c_k-\tfrac{5}{2}c_k^2)-\tfrac{1}{2}z\,c_k'^3c_k^2\ ,$$
$$a_{3,1}^{1}=a_{3,1}^{2}=-z\,,\ a_{3,2}^{1}=a_{3,2}^{2}=1\,,\ a_{3,3}^{1}=a_{3,3}^{2}=0\,,$$
$$b_{k,1}^{1}=v\,c_k^3(6c_k'^2+3c_k'+1)\,,\ b_{k,2}^{1}=-v\,c_k^3(3c_k'^2+c_k')\,,$$
$$b_{k,3}^{1}=\tfrac{1}{2}v\,c_k^3c_k'^2\,,\ b_{k,1}^{2}=v\,c_k'^3(6c_k^2+c_k+1)\,,\ b_{k,2}^{2}=v\,c_k'^3(3c_k^2+c_k)\,,$$
$$b_{k,3}^{2}=\tfrac{1}{2}v\,c_k'^3c_k^2\,,\ \ b_{3,1}^{1}=b_{3,1}^{2}=1\,,\ b_{3,k}^{1}=b_{3,k}^{2}=0\,,$$

for k=1,2 and $\bar{S}_i=(S_i^{(0)},S_i^{(1)},S_i^{(2)})^T$, $\bar{S}_{i-1}=(S_{i-1}^{(0)},S_{i-1}^{(1)},S_{i-1}^{(2)})^T$

$$\bar{S}_{i-m}=(S_{i-m}^{(0)},S_{i-m}^{(1)},S_{i-m}^{(2)})^T\,,\ \ \bar{S}_{i-m-1}=(S_{i-m-1}^{(0)},S_{i-m-1}^{(1)},S_{i-m-1}^{(2)})^T$$

And hence we get

$$W(z,v)M_i=G(z,v)M_{i-1} \tag{4.5}$$

where $M_i=(\bar{S}_i,\bar{S}_{i-m})^T$, $M_{i-1}=(\bar{S}_{i-1},\bar{S}_{i-m-1})^T$,

$$W(z,v)=\left[A_1\,|-B_1\right],\ G(z,v)=\left[A_2\,|-B_2\right].$$

Thus by definition, $z=\lambda h,\ v=qh$ belongs to the region of P-stability of the methods. It is clear that $(z,v)\in S_P$ if the eigenvalues $\mu_j(z,v),\ j=1(1)3$ of the generalized eigenvalue problem

$$\mu W\ (z,v).\underline{x}=G(z,v).\underline{x},\ \underline{x}\neq 0 \tag{4.6}$$

lie inside to the unit disc, i.e. if

$$|\mu_j(z,v)|<1,\ j=1(1)3. \tag{4.7}$$

Now, let $\det[\mu W(z,v)-G(z,v)]=0$ be the characteristic equation of (4.6).

Numerical experiments indicate that (4.7) is satisfied for various values c_1, c_2 listed in Table(2). In these cases the methods are P-stable, where the P-stability regions S_P were obtained numerically by determining $\{\,(z,v)\,,\,|\,v\,|<-\mathrm{Re}(z)\,\}$ according to $\det[e^{\varphi i}W(z,v)-G(z,v)]=0$ (cf.[7], Ch.2). On the other hand, the methods are P-stable for all $0.8028\leq c_1<c_2<1$. See some regions of P-Stability in the Fig.(1) for c_2=0.98, and different c_1.

Table(2): Some P-stability intervals for the methods

$0.55\leq c_1$	$0.978355\leq c_2<1$
$0.60\leq c_1$	$0.953197\leq c_2<1$
$0.65\leq c_1$	$0.924084\leq c_2<1$
$0.70\leq c_1$	$0.890401\leq c_2<1$

$0.75 \le c_1$	$0.851299 \le c_2 < 1$
$0.80 \le c_1$	$0.805586 \le c_2 < 1$
$0.8025 \le c_1$	$0.803099 \le c_2 < 1$
$0.8028 \le c_1$	$0.802800 < c_2 < 1$

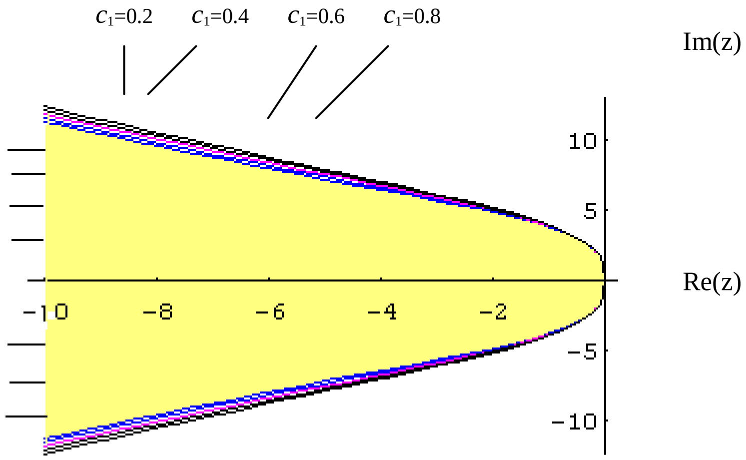

Fig.(1): Some regions of P-stability for c_2=0.98 and different c_1.

5-Numerical examples

In this section we give the numerical examples(1-5) to demonstrate the reliability, precision of the methods as compared to spectral method (SPC) [14], routine SPL [13] with *N* linear spline elements and routine HRKF4 [16]. The HRKF4 method is an adaptive fourth-order Runge-Kutta method with a fifth-order Hermite interpolation of the delayed variables. Here, δ indicates the absolute error norm in Tables (4-6). The examples were chosen because they exhibited difficulties characteristic of the delay-differential equations: a combination of stiffness and delay, and the nonsmooth or highly oscillatory character of the initial function. All computations are made with the computer of Turbo PASCAL 7.0 in double precision.

Example 1. The Ex.(1) is a single delay equation with a stiffness parameter (cf. [14])

$$y'(t) = Ay(t) + y(t - \frac{3\pi}{2}) - A\sin(t),$$

$$y(\theta) = e^{p\theta} + \sin(\theta), \qquad \theta \in [-\frac{3\pi}{2}, 0],$$

where $A = p - e^{-3\pi p/2}$.

The solution is given by $y(t) = e^{pt} + \sin(t)$.

For large negative values of *p*, the solution consists of a short transient of exponential decay, followed by a periodic sinusoidal oscillation. The parameter *p* also enters the delay equation exponentially; therefore, its effect on the stiffness of the equation is dramatic. In Table(4) we computed the absolute errors at different time levels, for *p* values of (-0.1, -1.0, -2.0).

Example 2. Here we consider the following system of four homogeneous delay-differential equations (cf. [14]):

$$y_1'(t) = y_3(t),$$
$$y_2'(t) = y_4(t),$$
$$y_3'(t) = -2n\, y_2(t) + (1+n^2)(-1)^n y_1(t-\pi),$$
$$y_4'(t) = -2n\, y_1(t) + (1+n^2)(-1)^n y_2(t-\pi).$$

The initial functions and solutions are given by

$$y_1(t) = \sin(t)\cos(nt),$$
$$y_2(t) = \cos(t)\sin(nt),$$
$$y_3(t) = y_1'(t),$$
$$y_4(t) = y_2'(t), \qquad t \in [-\pi, \infty).$$

Fig.(2) shows both the approximate solution by the proposed method and the exact solution for $h=\pi/100$ and $n=6$. In Table(5) we compare the absolute errors of the method and other methods.

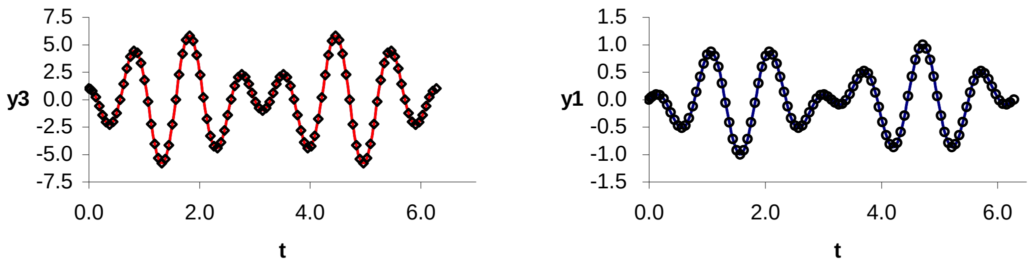

Fig.(2). Both the approximate solution and exact solution of example 2, for *n*=6, c_1=0.25, c_2=0.75.

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Example 3 [14, Ex.(3)]. We consider a degenerate system where $(1,-2,-1)^T y(t) = 0$ for $t \geq 2$ and all initial data $(\eta, \phi) \in R^n \times L^2$:

$$y'(t) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} y(t-1).$$

The solution is then given by

$$y(t) = \begin{cases} (1+2t-t^2, 1-t, 1)^{\mathrm{T}}, & t \in [0,1], \\ (2,\ 0, -2+4t-t^2)^{\mathrm{T}}, & t \in [1,2], \\ (2,\ 0,\ 2)^{\mathrm{T}}, & t \geq 2. \end{cases}$$

We note that the initial function is discontinuous at t=0 and the solution components, y_2 and y_3, have a jump discontinuity in the derivative at t=1. Table(6) clearly shows the high accuracy of the methods. Fig.(3) shows both the approximate solution and the exact solution for *h*=0.1.

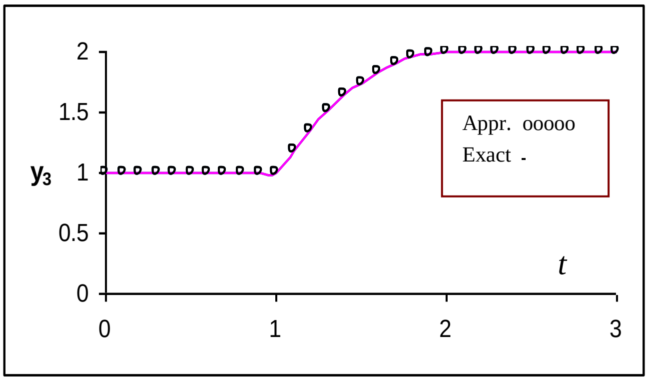

Fig.(3). Both the approximate solution and the exact solution of example 3, for c_1=0.25, c_2=0.75.

Example 4 [11, 15]: Artificial problem

$$y'(t) = 1 - y[\exp(1 - \frac{1}{t})], \qquad t \in [0.1, 10]$$

$$\phi(t) = \ln(t), \qquad t \in [0, 0.1]$$

The exact solution is *y*(*t*)=ln(*t*). Note the vanishing delay function as $t \to 1$. Table(3) shows our numerical results.

Example 5 [1]: Neutral delay differential equation.

$$y''(t) = \cos(t) - \frac{1}{2}y(t) + \frac{1}{2}y(t-\pi) - y'(t-\pi), \quad t \geq 0,$$

$$y(t) = 1, \quad -\pi \leq t \leq 0$$

The exact solution for this problem with the given initial function is:

$$y(t) = 1 - 2\cos(t) + 2\cos(\frac{\sqrt{3}}{3}t), \quad for\ t \in [0, \pi],$$

Table(7) shows comparisons between the proposed method and cubic approximations and deficient spline approximations of order 3.

Table(3): Numerical results for Ex.(4), with *h*=0.02, c_1=0.25, c_2=0.75.

t	**Absolute error of the method**
0.2	2.1E-13
1	4.7E-13
2	7.4E-13
4	9.5E-13
6	1.0E-12
8	2.1E-12
10	2.2E-12

Table(4): Test results for Ex.(1), with $h = \pi/40$, c_1=0.25, c_2=0.75

P	Time	δ_{SPC}^{8} [14]	δ_{SPL}^{200} [14]	δ_{HRF} [14]	**Present Method**
-0.1	$3\pi/4$	2.6E-6	6.2E-6	2.1E-4	**6.0E-12**
	$3\pi/2$	7.9E-8	1.2E-5	4.7E-5	**9.3E-12**
	$9\pi/4$	1.0E-5	2.5E-5	1.7E-4	**1.8E-11**
	3π	3.1E-7	3.1E-5	1.3E-4	**2.9E-11**
	$15\pi/4$	8.4E-7	2.0E-5	4.5E-5	**4.4E-11**
P	Time	δ_{SPC}^{10}	δ_{SPL}^{200}	δ_{HRF}	**Present Method**
-1.0	$3\pi/4$	8.3E-8	2.5E-7	3.1E-7	**9.8E-15**
	$3\pi/2$	7.6E-7	6.6E-11	2.1E-7	**2.2E-14**
	$9\pi/4$	1.5E-8	2.9E-07	1.7E-7	**3.6E-14**
	3π	4.2E-7	4.1E-07	9.8E-8	**7.1E-14**
	$15\pi/4$	2.0E-7	2.9E-07	9.8E-7	**1.5E-13**
P	Time	δ_{SPC}^{16}	δ_{SPL}^{200}	δ_{HRF}	**Present Method**
-2.0	$3\pi/4$	1.3E-10	1.1E-8	1.5E-7	**1.5E-15**
	$3\pi/2$	1.1E-9	2.2E-9	1.6E-7	**3.3E-15**
	$9\pi/4$	2.1E-10	2.1E-9	1.6E-7	**5.2E-15**
	3π	1.1E-09	3.9E-9	6.4E-9	**7.5E-15**
	$15\pi/4$	2.1E-10	2.6E-9	Failed	**9.7E-15**

Table(5): Test results for Ex.(2), with $h = \pi/100$, c_1=0.25, c_2=0.75.

n	Time	δ_{SPC}^{8} [14]	δ_{SPL}^{2000} [14]	δ_{HRF} [14]	**Present Method**
1	$\pi/2$	2.1E-4	4.7E-6	6.8E-6	**3.1E-13**
	π	5.7E-7	9.6E-6	1.2E-5	**3.8E-13**
	$3\pi/2$	2.0E-4	9.0E-6	1.2E-5	**5.1E-13**
	2π	2.8E-6	1.0E-5	1.6E-5	**6.3E-13**
n	Time	δ_{SPC}^{10}	δ_{SPL}^{2000}	δ_{HRF}	**Present Method**
2	$\pi/2$	2.8E-4	2.1E-7	7.6E-8	**7.3E-12**
	π	7.5E-7	5.0E-7	5.8E-7	**9.1E-12**
	$3\pi/2$	3.0E-4	1.2E-5	1.3E-5	**3.8E-11**
	2π	5.1E-6	2.7E-4	3.1E-4	**1.4E-10**
n	Time	δ_{SPC}^{18}	δ_{SPL}^{40000}	δ_{HRF}	**Present Method**
6	$\pi/2$	3.6E-4	3.3E-6	6.6E-11	**6.7E-11**
	π	1.9E-9	7.7E-4	1.5E-08	**5.4E-10**
	$3\pi/2$	4.0E-4	1.8E-1	3.5E-06	**2.0E-08**
	2π	1.0E-4	41E-2	8.1E-04	**1.0E-06**

Table(6): Test results for Ex.(3), with *h*=0.1, c_1=0.25, c_2=0.75.

Time	δ^2_{SPC} [14]	δ^1_{SPL} [14]	δ_{HRF} [14]	**Present Method**
1.0	2.2E-16	0.0	7.5E-16	**3.0E-17**
2.0	1.0E-15	0.0	4.5E-12	**1.1E-16**
3.0	2.1E-15	0.0	1.1E-11	**1.3E-16**

Table(7): Absolute errors for the solution of Ex.(5).

k	deficient spline approx.[1] of order 3, for *h*=22/1400	Cubic Approx. method[1] for *h*=22/1400	**Present method** for $h = \pi/50$
0	3.958656358E-04	2.0161678549E-07	**6.7131677348E-11**
1	1.5321755618E-03	1.3896369637E-06	**2.7210381301E-10**
2	3.3129796484E-03	3.2549742173E-06	**6.1921654869E-10**
3	5.6547611457E-03	5.7729739638E-06	**1.1100225014E-09**
4	8.4848242204E-03	9.2825812317E-06	**2.5255069195E-09**
5	1.1739892165E-02	1.3213615603E-05	**3.4499927112E-09**

6 Conclusions

- The main contribution of this paper is the development and analysis of a class of spline collocation methods for solving delay-differential equations. The developed methods depend on C^2-spline collocation schemes determined by quintic Hermite interpolation with two parameters $c_1, c_2 \in (0,1)$, $c_1 \neq c_2$.
- The methods possess convergence rate of order six when $58 - 57(c_1 + c_2) + 55c_1c_2 = 0$ (see, Table(1)), in the remaining cases the order is five. Moreover, the methods are P-stable for $0.8028 \le c_1 < c_2 < 1$, and increase regions of P-stability when $c_2 = 0.98,\ c_1 \to 1^-$ (see, Fig. (1)).
- Numerical results illustrating the behavior of the methods when faced with some difficult problems are presented and the numerical results are compared to those obtained by other methods (see, Examples 1-4).
- The comparisons of our numerical results with other methods show that our results are more accurate (see, Tables 4-7).
- Our methods if applied to delay-differential equations are successful for solving problems, which have oscillatory solutions; (see, Fig.(2), ex.(2)).
- Another advantage is that these methods attain the same order of accuracy for delay-differential equations as they do for ordinary differential equations because spline approximation is directly used to interpolate delay function.

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