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Numerical Treatment of The Falkner–Skan Equation Using Spline Function Approximations

Dr. Suliman. M. Mahmoud*

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□ ABSTRACT □

This paper presents an iterative method based on spline function approximations for the numerical solution of the Falkner–Skan equation (FSE) over a semi-infinite interval. This technique will transform the FSE into two initial value problems, so the solution of FSE will be reduced from the interval $[0,\infty[$ into [0,1]. Spline approximations are applied directly to the FSE without its reducing into a system of first-order differential equations, thus, the algorithm of spline method has a computational cost that is cost-effective. The spline solution of the FSE is existent and unique, and the convergence analysis for the spline method applied to the FSE is discussed. Numerical results are compared with those obtained by previous methods under various instances of the FSE. The comparisons show the accuracy and efficiency of the presented methodology.

Key Words: Falkner–Skan equation, Nonlinear boundary value problems, Initial-value problems, Spline function approximations, collocation points, Convergence Analysis.

***Associate Professor, Department of Mathematics, Faculty of Science, Tishreen University, Lattakia, Syria.**
E-mail: Suliman_mmn@yahoo.com.

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معالجة عددية لمسألة فولكنر–سكان باستخدام تقريبات دالة شرائحية

الدكتور سليمان محمد محمود*

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□ ملخّص □

تم تقديم طريقة تكرارية مبنية على تقريبات دالة شرائحية لحل معادلة فولكنر–سكان فوق مجال شبه لانهائي. هذه التقنية ستحول مسألة القيمة الحدية المطروحة التي حلها في المجال $]0,\infty[$ إلى مسألتين من مسائل القيم الابتدائية حليهما في المجال $[0, 1]$. نطبق التقريبات الشرائحية مباشرة على المسألة دون تخفيضها إلى جملة معادلات تفاضلية من المرتبة الأولى، وهذا بالتأكيد سيقلل من الكلفة الحسابية لخوارزمية الطريقة. الطريقة المقترحة تمكّننا دوماً من إيجاد الحل التقريبي ومشتقاته حتى المرتبة الثالثة للمسألة عند أي نقطة من مجال الحل. كما تم إثبات أن الحل الشرائحي للمسألة موجود بشكل وحيد، بالإضافة إلى مناقشة لتحليل تقارب الطريقة.

أخيراً نقدم الحلول العددية للمسألة وفق مختلف حالاتها الخاصة المعروفة وفوق مجالات متنوعة، حيث تشير النَتائِج العددية إلى دقة وفعالية الطرائق المقترحة مقارنة مع نتائج بعض الطرائق الأخرى.

الكلمات المفتاحية: معادلة فولكنر– سكان، مسائل القيم الحدية غير الخطية، مسائل القيم الابتدائية، تقريبات دالة شرائحية، نقاط تجميعية، تحليل التقارب.

* أستاذ مساعد – قسم الرياضيات – كلية العلوم– جامعة تشرين– اللاذقية– سورية
. E-mail: Suliman_mmn@yahoo.com.

Introduction:

Generally, no closed-form solutions are available for nonlinear two-point boundary value problems (BVPs). Numerical solutions of these problems have always been of interest for scientists and engineers. The well-known nonlinear third-order Falkner-Skan equation is much more challenging since it is a BVP depicted on an infinite interval. This problem of laminar boundary layer resulting from the flow of an incompressible fluid past a semi-infinite wedge is of considerable practical and theoretical interest. The solutions of the Falkner-Skan equation are similarity solutions of the two-dimensional incompressible laminar boundary layer equations. Due to the appearance of irregular boundaries, shock waves, boundary layers, derivative boundary conditions, etc., the solutions so obtained have in many cases been unsatisfactory because of poor resolution spurious oscillations, and excessive computer time storage. The Falkner-Skan equation is given by [1-14]

$$f'''(\eta)+\alpha f(\eta)f''(\eta)+\beta[1-(f'(\eta))^2]=0\ ,\quad 0<\eta<\infty \qquad (1.1)$$

subject to the boundary conditions

$$f(\eta)=0\text{, as } \eta=0\ , \qquad (1.2a)$$

$$f'(\eta)=0\text{, as } \eta=0\ , \qquad (1.2b)$$

$$f'(\eta)=1\text{, as } \eta\to\infty\ , \qquad (1.2c)$$

where, f' the fluid velocity, is a function of η, α is assumed constant and β is a measure of the pressure gradient. The prime denotes differentiation with respect to η. We can find easily that the solution of equation (1.1) satisfies the asymptotic condition:

$$f''(\eta)\to 0 \text{ as } \eta\to\infty \qquad (1.3)$$

Special Cases of the Falkner–Skan Equation (FSE) [9, 13, 14]:

- **Case 1**: If $\alpha=1/2$ or $\alpha=1$ and $\beta=0$, the FSE is called a *Blasius flow*. This Blasius flow gets FSE in the study of a laminar boundary layer along a thin flat plate.
- **Case 2:** If $\alpha=0,\ \beta=1$, the FSE is called a *Pohlhausen flow* .
- **Case 3:** If $\alpha=1,\ \beta=1/2$, the FSE is called a *Homann flow*.
- **Case 4:** If $\alpha=1,\ \beta=1$, the FSE is known as the *Hiemenz flow*.
- **Case 5**: Other cases are resulted from the FSE when **(** $\alpha=1,\ \beta\in[-0.1988,\ \infty[$).

Numerical techniques for the solution of the Falkner–Skan equation are mainly based on : finite differences [1,2,11], shooting [5,13], quasilinearization method [14], Chebyshev spectral method [3,8,4], differential transformation method [7], Homotopy analysis method [12] and rational scaled generalized Laguerre function collocation method for solving the Blasius equation[9].

In any case the far field boundary condition is a problem, where the correct value of unknown shear stress f''_w at the wall must be found which ensures an asymptotic approach of the velocity values of f' at infinity to unity the well-known matching condition of viscous solution (near-wall) to the inviscid solution. Such techniques are termed as "shooting method".

1. Importance and Aim of This Work:

It is well known that the Falkner–Skan equation is very difficult to be solved when there are many cases, for this reason, the numerical solutions are very important. We purpose to develop a numerical method based on spline function approximations to solve the Falkner–Skan equation over a semi-infinite interval. The algorithm of spline method has a computational cost that is cost-effective. The proposed method will enables to approximate the solution and its derivatives of the problem at every point of the range of integration. The approximate solutions and their derivatives are illustrated for special flows such as: Blasius, Pohlhausen, Homann and Hiemenz flows over various intervals. The local errors and the rate of convergence are computed for present spline method. The approximate solutions of the wall shear stress of Falkner–Skan equation over various intervals are showed.

2. Methodology:

Theoretical part: The Falkner–Skan equation and its solution are transformed over the interval $[0,\infty[$ into two IVPs and their solutions in the interval [0, 1]. After that, Spline approximations are applied directly to approximate the solution and its derivatives up to third-order of the problem at every point of the range of integration. The spline approximation solution of the Falkner–Skan problem is existent and unique. The convergence analysis and the rate of convergence for the spline method applied to the problem are discussed. Finally, an iterative algorithm is proposed for solving Falkner–Skan problem.

Practical part: Numerical results for various instances are compared with those obtained by others. The comparisons show the accuracy, robustness and efficiency of the presented methodology. The computations are accomplished by using *Mathematica* Version 5 and Turbo Pascal in double precision.

3. Method of Solution:

With the change of variable $y=\dfrac{f}{\eta_\infty}$ and under the transformation $x=\dfrac{\eta}{\eta_\infty}$, the Falkner–Skan equation (1.1) is transformed to

$$y'''(x)+\eta_\infty^2\,\alpha\,y(x)\,y''(x)+\eta_\infty^2\,\beta[1-(y'(x))^2]=0\ ,\quad x\in[0,1], \tag{1.4}$$

where

$$\frac{dy}{dx}=\frac{1}{\eta_\infty}\frac{df}{d\eta}\frac{d\eta}{dx}=\frac{df}{d\eta}\ ,\qquad \frac{d2y}{dx^2}=\frac{1}{\eta_\infty}\frac{d2f}{d\eta^2}\ ,\qquad \frac{d3y}{dx^3}=\frac{1}{\eta_\infty^2}\frac{d3f}{d\eta^3}\ .$$

The boundary conditions (1.2) are transformed to

$$y(0)=y'(0)=0 \text{ and } y'(1)=1, \tag{1.5}$$

The asymptotic boundary condition (1.3) is transformed to

$$y''(1)=0 \tag{1.6}$$

The shooting technique for the nonlinear third-order boundary-value problem (1.4) is similar to the linear case, except that the solution to a nonlinear problem cannot be simply expressed as a linear combination of the solutions to two initial-value problems [15]. Instead, it needs to use the solutions to a sequence of initial-value problems of the form:

$$y'''(x,t)+\eta_\infty^2\,\alpha\,y(x,t)\,y''(x,t)+\eta_\infty^2\,\beta[1-(y'(x,t))^2]=0\ , \tag{1.7}$$

subject to the initial conditions

$$y(0,t)=y'(0,t)=0 \text{ and } y''(0,t)=t\,, \tag{1.8}$$

here the solution depend on both x and *t*.

We do this choosing the parameters $t=t_k$ in a manner to ensure that

$$\lim_{k\to\infty} y'(1,\ t_k)=y'(1)=1 \tag{1.9}$$

where $y(x,t_k)$ denotes the solution of the initial-value problem (1.7)(1.8) with $t=t_k$, and $y(x)$ denotes the solution to the boundary value problem (1.4)-(1.5).

We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (0,0) and along the curve described by the solution to the initial-value problem (1.7) with initial conditions $y(0,t)=y'(0,t)=0$ and $y''(0,t)=t_0$. If $y'(x,t_0)$ is not sufficiently close to 1, we attempt to correct our approximation by choosing another elevation t_1 and so on, until $y'(x,t_k)$ is sufficiently close to 1.

If $y(x,t)$ denotes the solution to the initial-value problem (1.7)-(1.8), the problem is to determine t so that

$$y'(1,t)-1=0 \qquad (1.10)$$

Since this is a nonlinear equation, we shall use Newton's iteration method to generate the sequence $\{t_k\}$, only one initial value, t_0, is needed, however, the iteration has the form:

$$t_k = t_{k-1} - \frac{y'(1,t_{k-1})-1}{\frac{d}{dt}y'(1,t_{k-1})}, \quad k=1,2,\ldots \qquad (1.11)$$

and requires the knowledge of $\frac{d}{dt}y'(1,t_{k-1})$.

Still this presents a difficulty, since an explicit representation for $y'(1,t)$ is not known; where we know only the values $y'(1,t_0), y'(1,t_1),\ldots, y'(1,t_{k-1})$. For this reason, we take the partial derivative of problem (1.7) . This implies that

$$\frac{\partial}{\partial t}y'''(x,t) = \frac{\partial}{\partial t}\{-\eta_\infty^2\alpha\, y(x,t)\,y'' - \eta_\infty^2\beta[1-(y'(x,t))^2\,]\}$$

$$= \frac{\partial}{\partial x}\{-\eta_\infty^2\alpha\, y\,y'' - \eta_\infty^2\beta[1-(y')^2\}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\{-\eta_\infty^2\alpha\, y\,y'' - \eta_\infty^2\beta[1-(y')^2\}\frac{\partial y}{\partial t} +$$

$$\frac{\partial}{\partial y'}\{-\eta_\infty^2\alpha\, y\,y'' - \eta_\infty^2\beta[1-(y')^2\,]\}\frac{\partial y'}{\partial t} + \frac{\partial}{\partial y''}\{-\eta_\infty^2\alpha\, y\,y'' - \eta_\infty^2\beta[1-(y')^2\}\frac{\partial y''}{\partial t}$$

Since *x* and *t* are independent, then we have

$$\begin{aligned}\frac{\partial}{\partial t}y'''(x,t) &= -\eta_\infty^2\alpha\, y''(\eta,t)\frac{\partial y}{\partial t}(x,t) + 2\eta_\infty^2\beta\, y'(x,t)\frac{\partial y'}{\partial t}(x,t)\\ &\quad -\alpha\eta_\infty^2\, y(x,t)\frac{\partial y''}{\partial t}(x,t), \quad x\in[0,1],\end{aligned} \qquad (1.12)$$

the initial conditions (1.8) are given by:

$$\frac{\partial}{\partial t}y(0,t)=0,\ \frac{\partial}{\partial t}y'(0,t)=0\ ,\ \frac{\partial}{\partial t}y''(0,t)=1. \qquad (1.13)$$

By using $U(x,t)$ to denote $\frac{\partial}{\partial t}y(x,t)$ and assume that the order of differentiation of *x* and *t* can be reversed, equations (1.12)-(1.13) transform to the initial-value problem:

$$\begin{aligned}&U'''(x,t)+\alpha\eta_\infty^2 y''(x,t)U(x,t)-2\eta_\infty^2\beta\, y'(x,t)U'(x,t)+\eta_\infty^2\alpha\, y(x,t)U''(x,t)=0,\\ &U(0,t)=0\ \ ,U'(0,t)=0\quad and\quad U''(0,t)=1\end{aligned} \qquad (1.14)$$

Now, it is required to solve the two initial-value problems (1.7)-(1.8) and (1.14) for each iteration by using the iteration relation (1.11), which takes the form:

$$t_k = t_{k-1} - \frac{y'(1,\ t_{k-1}) - 1}{U'(1,\ t_{k-1})}, \quad k=1,2,\dots \tag{1.15}$$

Spline Approximations:

Consider the uniform grid partition $\Delta :\equiv \{0 = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_N = 1\}$ of the interval $[0,1]$ with mesh size $h = 1/N$ and grid points $x_k = k\,h,\ k = 0,...N$. The Spline function approximation $S(x)$ to the function f at the grid points is given by the piecewise expression:

$$S(x) = \begin{cases} S_0(x), & if\ \ x \in [x_0, x_1], \\ S_k(x), & if\ \ x \in [x_k, x_{k+1}],\ k = 1,...,N-2, \\ S_{N-1}(x), & if\ \ x \in [x_{N-1}, x_N], \end{cases} \tag{2.1}$$

where

$$S_k(x) = \sum_{i=0}^{4} \frac{(x - x_k)^i}{i!} S_k^{(i)} + \sum_{i=5}^{7} \frac{(x - x_k)^i}{i!} C_{k,i-4},\ x \in [x_k, x_{k+1}],\ k = 0,...,N-1, \tag{2.2}$$

By applying the Spline approximation (2.2) and its derivatives with respect to x, $S'_k(x)$, $S''_k(x)$, $S'''_k(x)$ on three collocation points $x_{k+z_j} = x_k + z_j\,h$, (j=1,2,3), into Falkner–Skan equation (1.7)-(1.8), in each subinterval $I_k = [x_k,\ x_{k+1}]$, k=0(1)N-1, we have

$$S'''_k(x_{k+z_j}) + \alpha\eta_\infty^2 S_k(x_{k+z_j}) S''_k(x_{k+z_j}) + \beta\eta_\infty^2 [1 - (S'_k(x_{k+z_j}))^2] = 0,\ j=1,2,3 \tag{2.3a}$$

with the initial conditions

$$S_0(0) = S'_0(0) = 0,\ \ S''_0(0) = t, \tag{2.3b}$$

where $S_0(0) = y(0,t) = S'_0(0) = y'(0,t) = 0$ and $S''_0(0) = y''(0,t) = t$, the other coefficients $S_0^{(m)}(0) = y^{(m)}(0,t)$, for $m \geq 3$ are determined by the derivation of equation (1.7), where

$$S_k(x_{k+z_j}) = \sum_{i=0}^{4} \frac{(h z_j)^i}{i!} S_k^{(i)} + \sum_{i=5}^{7} \frac{(h z_j)^i}{i!} C_{k,i-4},\ \ x_{k+z_j} \in [x_k, x_{k+1}],\ \ j = 1,2,3, \quad k = 0,...,N-1, \tag{2.4}$$

The first three coefficients $C_{k,1}, C_{k,2}, C_{k,3}$ are computed from the nonlinear system (2.3a) by using the initial value conditions(2.3b) if k=0, or from the previous step if k>1.

Similarly, we need to find the approximate spline solution to the initial-value problem (1.14), to do so, we put spline approximation:

$$S_{u,k}(x) = \sum_{i=0}^{4} \frac{(x - x_k)^i}{i!} S_{u,k}^{(i)} + \sum_{i=5}^{7} \frac{(x - x_k)^i}{i!} \tilde{C}_{k,i-4},\ x \in [x_k, x_{k+1}],\ k = 0,...,N-1, \tag{2.5}$$

by applying (2.5) into (1.14) to be satisfied at collocation points $x_{k+z_j} = x_k + z_j\,h$, j=1(1)3, we get:

$$\begin{aligned} &S'''_{u,k}(x_{k+z_j}) + \alpha\eta_\infty^2 S''_k(x_{k+z_j}) S_{u,k}(x_{k+z_j}) + \alpha\eta_\infty^2 S_k(x_{k+z_j}) S''_{u,k}(x_{k+z_j}) \\ &\quad - 2\beta\eta_\infty^2 S_k(x_{k+z_j}) S_{u,k}(x_{k+z_j}) = 0 \end{aligned} \tag{2.6a}$$

with the initial values:

$$S_{u,0}(0)=S'_{u,0}(0)=0,\ \ S''_{u,0}(0)=1, \tag{2.6b}$$

where

$$S_{u,k}(x_{k+z_j})=\sum_{i=0}^{4}\frac{(hz_j)^i}{i!}S_{u,k}^{(i)}+\sum_{i=5}^{7}\frac{(hz_j)^i}{i!}\tilde{C}_{k,i-4}, x_{k+z_j}\in[x_k,x_{k+1}],\ \ j=1,2,3,\quad k=0,...,N-1, \tag{2.7}$$

and

$$0<z_1<z_2<z_3=1 \tag{2.8}$$

The first three coefficients $\tilde{C}_{k,1},\tilde{C}_{k,2},\tilde{C}_{k,3}$ are computed from the nonlinear system (2.6a) by using the initial value conditions (2.6b) if k=0, or from the previous step if k>1.

Substituting $S'_{N-1}(x_N,t_{k-1}), S'_{U,N-1}(x_N,t_{k-1})$ are obtained by solving (2.3) and (2.6) into the iteration relation (1.15), we have

$$t_k=t_{k-1}-\frac{S'_{N-1}(x_N,t_{k-1})-1}{S'_{U,N-1}(x_N,t_{k-1})},\quad k=1,2,\ldots \tag{2.9}$$

1. A unique Solution

As previous, the Falkner–Skan equation can be written in the following nonlinear form:

$$f'''(x)=F(x,f(x),f'(x),f''(x)),\ x\in[0,b] \tag{2.10a}$$

$$f(0)=a_0, f'(0)=a_1 \text{ and } f''(0)=a_2 \tag{2.10b}$$

and suppose that $F:[0,b]\times C[0,b]\times C^1[0,b]\times C^2[0,b]\to R$ is an enough smooth function satisfying the following Lipschitz condition in respect to the last three arguments:

$$|F(x,y_1,y_2,y_3)-F(x,u_1,u_2,u_3)|\le L(|y_1-u_1|+|y_2-u_2|+|y_3-u_3|),\quad \forall\ (x,y_1,y_2,y_3),(x,u_1,u_2,u_3)\in[0,b]\times R^3 \tag{2.11}$$

where the constant L is called a Lipschitz constant for F.

These conditions assure the existence of a unique solution $f(x)$ of problem (2.10).

By applying the Spline approximations (2.2) and its derivatives into the problem (2.10), using three collocation points $x_{k+z_j}=x_k+z_j h$, (j=1,2,3), we obtain the system

$$S_k'''+(hz_j)S_k^{(4)}+\frac{(hz_j)^2}{2!}C_{k,1}+\frac{(hz_j)^3}{3!}C_{k,2}+\frac{(hz_j)^4}{4!}C_{k,3}= F(x_{k+z_j},S(x_{k+z_j}),S'(x_{k+z_j}),S''(x_{k+z_j}))\ ,\ \ j=1,2,3,\quad k=0,...,N-1, \tag{2.12a}$$

$$S_0(0)=a_0, S_0'(0)=a_1 \text{ and } S_0''(0)=a_2 \tag{2.12b}$$

We rewrite (2.12a) in the matrices formula:

$$A\overline{C}_k=\overline{S}_k+\overline{F}_k \tag{2.13}$$

where

$$A=\begin{bmatrix} \frac{z_1^2h^2}{2!} & \frac{z_1^3h^2}{3!} & \frac{z_1^4h^2}{4!} \\ \frac{z_2^2h^2}{2!} & \frac{z_2^3h^2}{3!} & \frac{z_2^4h^2}{4!} \\ \frac{h^2}{2!} & \frac{h^2}{3!} & \frac{h^2}{4!} \end{bmatrix},\ \overline{C}_k=\begin{bmatrix} C_{k,1} \\ hC_{k,2} \\ h^2C_{k,3} \end{bmatrix},\ \overline{S}_k=\begin{bmatrix} -S_k^{(3)}-hz_1S_k^{(4)} \\ -S_k^{(3)}-hz_2S_k^{(4)} \\ -S_k^{(3)}-hS_k^{(4)} \end{bmatrix},\ \overline{F}_k=\begin{bmatrix} F_{k+z_1} \\ F_{k+z_2} \\ F_{k+1} \end{bmatrix},$$

$$F_{k+z_j}=F(x_{k+z_j},S(x_{k+z_j}),S'(x_{k+z_j}),S''(x_{k+z_j})),\ j=1,2,3.$$

Theorem 1: Suppose that $F\in C^2([0,b]\times\Re\times\Re\times\Re)$ satisfies Lipschitz condition, and if

$$h<\frac{933120}{279073L} \qquad (2.14)$$

then the spline approximation solution S(x) exists and is uniquely defined for $z_1=1/2,\ z_2=2/3, z_3=1$, where L is a Lipschitz constant for F.

Proof. It is sufficient to prove that the vector $\overline{C}_k$ can be uniquely determined for arbitrary given $\overline{S}_k$. Let $\overline{C}_k^1,\overline{C}_k^2\in R^3$, then using $\|\,.\,\|_1$ from (2.13), we have

$$\overline{C}_k^1=A^{-1}\underline{S}_k+A^{-1}\overline{F}_k^1 \text{ and } \overline{C}_k^2=A^{-1}\underline{S}_k+A^{-1}\overline{F}_k^2$$

Thus $\overline{C}_k^1$ and $\overline{C}_k^2$ can be written in the form

$$\overline{C}_k^1=\overline{Q}_k(C_{k,1}^1,C_{k,2}^1,C_{k,3}^1,h) \text{ and } \overline{C}_k^2=\overline{Q}_k(C_{k,1}^2,C_{k,2}^2,C_{k,3}^2,h)$$

Applying $\|\,.\,\|_1$, Lipschitz condition and using Mathematica, we get

$$\|\overline{Q}_k(\overline{C}_k^1)-\overline{Q}_k(\overline{C}_k^2)\|=\|(A^{-1}\underline{S}_k+A^{-1}\overline{F}_k^1)-(A^{-1}\underline{S}_k+A^{-1}\overline{F}_k^2)\|$$

$$=\|A^{-1}\|\,.\,\|\overline{F}_k^1-\overline{F}_k^2\|\le$$

$$\|A^{-1}\|\,.\{L_1(\frac{307}{1296}h^3+\frac{1633}{31104}h^4+\frac{9043}{933120}h^5)\,|C_{k,1}^1-C_{k,1}^2|+$$

$$L_2(\frac{1633}{31104}h^4+\frac{9043}{933120}h^5+\frac{51481}{33592320}h^6)\,|C_{k,2}^1-C_{k,2}^2|+$$

$$L_3(\frac{9043}{933120}h^5+\frac{51481}{33592320}h^6+\frac{298507}{1410877440}h^7)\,|C_{k,3}^1-C_{k,3}^2|\}$$

$$\le\|A^{-1}\|\,.L_4(\frac{307}{1296}h^3+\frac{1633}{31104}h^4+\frac{9043}{933120}h^5).$$

$$\{|C_{k,1}^1-C_{k,1}^2|+|C_{k,2}^1-C_{k,2}^2|+|C_{k,3}^1-C_{k,3}^2|\}$$

$$\le\frac{1}{h^2}L(\frac{279073}{933120}h^3)\{|C_{k,1}^1-C_{k,1}^2|+|C_{k,2}^1-C_{k,2}^2|+|C_{k,3}^1-C_{k,3}^2|\}$$

$$\le L(\frac{279073}{933120}h).\{\,|C_{k,1}^1-C_{k,1}^2|+|C_{k,2}^1-C_{k,2}^2|+|C_{k,3}^1-C_{k,3}^2|\}\ ,$$

where

$L_4=\max(L_1,\ L_2,L_3)$, $H_1=\max(H_1,\ H_2,H_3)$, for all $h\in]0,1[$, and

$$H_1=(\frac{307}{1296}h^3+\frac{1633}{31104}h^4+\frac{9043}{933120}h^5),$$

$$H_2=(\frac{1633}{31104}h^4+\frac{9043}{933120}h^5+\frac{51481}{33592320}h^6),$$
$$H_3=(\frac{9043}{933120}h^5+\frac{51481}{33592320}h^6+\frac{298507}{1410877440}h^7),$$
$$H_1\le\frac{279073}{933120}h^3\text{, for all } h\in]0,1[,$$
$$\|A^{-1}\|=\frac{1}{h^2}\,const,\; L=const.L_4\ .$$

Thus, the function $\overline{Q}_k$ defines a contraction mapping if $hL\frac{279073}{933120}<1$ which satisfies (2.14). Hence there exists a unique $\overline{C}_k$ that satisfies $\overline{C}_k=\overline{Q}_k(C_{k,1},C_{k,2},C_{k,3},h)$ which may be found by iterations, $\overline{C}_k^{\,p+1}=\overline{Q}_k\,(\overline{C}_k^{\,p},,h)$, p=0,1,2,… .

The proof of the Theorem 1 is now complete.

Convergence Analysis:

We assume that $y(x)\in C^8[0,1]$, the unique solution of the Falkner–Skan equation and S(x) be a spline approximation to y(x), also $T=(\overline{\tau}_k)$ is a 3-dimensional column vector. Here, the vector $\overline{\tau}_k$ is the local truncation error.

Applying the Spline approximation S(x) on three collocation points $x_{k+z_j}=x_k+z_j\,h$, (j=1,2,3), we approximately put $y(x_{k+z_j})\cong S(x_{k+z_j})$, and $y^{(m)}(x_k)=y_k^{(m)}\cong S^{(m)}(x_k)$, ($m$=0,…,4), k=0,…,N, for $z_1=1/2,\ z_2=2/3, z_3=1$, we obtain the local truncation error formula:

$$\overline{\tau}_k=\mathrm{M}\overline{C}_k-\overline{\Psi}_k\,, \tag{3.1}$$

where

$$\overline{\Psi}_k=\begin{bmatrix} y_k+\frac{h}{2}y'_k+\frac{h^2}{8}y''_k-\frac{h^3}{48}y'''_k+\frac{h^4}{384}y_k^{(4)}-y(x_k+\frac{1}{2}h)\\ y_k+\frac{2h}{3}y'_k+\frac{2h^2}{9}y''_k-\frac{4h^3}{81}y'''_k+\frac{2h^4}{243}y_k^{(4)}-y(x_k+\frac{2}{3}h)\\ y_k+h\,y'_k+\frac{h^2}{2}y''_k-\frac{h^3}{6}y'''_k+\frac{h^4}{24}y_k^{(4)}-y(x_k+h)\end{bmatrix},$$

$$\mathrm{M}=\begin{bmatrix} (\frac{1}{2^5})\frac{h^5}{5!} & (\frac{1}{2^6})\frac{h^5}{6!} & (\frac{1}{2^7})\frac{h^5}{7!}\\ (\frac{2}{3})^5\frac{h^5}{5!} & (\frac{2}{3})^6\frac{h^5}{6!} & (\frac{2}{3})^7\frac{h^5}{7!}\\ \frac{h^5}{5!} & \frac{h^5}{6!} & \frac{h^5}{7!}\end{bmatrix},\quad \overline{C}_k=\begin{bmatrix}C_{k,1}\\ hC_{k,2}\\ h^2C_{k,3}\end{bmatrix}$$

On the other hand, from the system (2.13), we get

$$\overline{C}_k=A^{-1}\hat{Y}_k+A^{-1}\hat{F}_k \tag{3.2}$$

where

$$Det(A)=\frac{h^6}{93312}\neq 0,\quad A^{-1}=\begin{bmatrix}\frac{64}{h^2} & \frac{-81}{h^2} & \frac{4}{h^2}\\ \frac{-480}{h^2} & \frac{729}{2h^2} & \frac{-42}{h^2}\\ \frac{1152}{h^2} & \frac{-972}{h^2} & \frac{144}{h^2}\end{bmatrix}.$$

$$\hat{Y}_k=\begin{bmatrix}-y_k^{(3)}-\frac{h}{2}y_k^{(4)}\\ -y_k^{(3)}-\frac{2h}{3}y_k^{(4)}\\ -y_k^{(3)}-h\,y_k^{(4)}\end{bmatrix},\quad \hat{F}_k=\begin{bmatrix}y'''(x_{k+z_1})\\ y'''(x_{k+z_2})\\ y'''(x_{k+1})\end{bmatrix}.$$

Using Taylor's expansions for the functions $y^{(m)}(x), m=0,...,4$ about x_k, in the relation (3.2) and substituting into (3.1), we get the local truncation error at the *k*th step as follows:

$$\bar{\tau}_k=\mathrm{M}(A^{-1}\hat{Y}_k+A^{-1}\hat{F}_k)-\overline{\Psi}_k=\begin{bmatrix}\frac{189}{10!.128}y^{(8)}(x_k)\\ \frac{869}{10!.234}y^{(8)}(x_k)\\ \frac{1}{7!.60}y^{(8)}(x_k)\end{bmatrix}h^8\equiv \mathrm{O}(h^8),k=0,1,..N \qquad (3.3)$$

where

$$y(x)=\sum_{i=0}^{8}\frac{(x-x_k)^i}{i!}y^{(i)}(x_k)+O(h^9),\ x\in[x_k,x_{k+1}],$$

Note from the relation (3.3) that the proposed Spline method is exact for expansions of the solution of degree ≤ 7, hence we have $\|T\|_\infty=N.O(h^8)\equiv O(h^7)$.

Consequently, we have obtained the following: Let $y\in C^8[0,1]$ be Lipschitz continuous, then the spline approximation $S(x)$ converges to the solution $y(x)$ of the Falkner–Skan problem as $h\to 0$ for $z_1=1/2$, $z_2=2/3$, $z_3=1$and

$$\lim_{h\to 0}S^{(m)}(e)=y^{(m)}(e),\ m=0,...,4,\ \ e=0,1. \qquad (3.4)$$

Furthermore, the convergence order is fifth, i.e., we have

$$|y^{(m)}(x)-S^{(m)}(x)|<C_m h^{8-m},\ m=0,...,3. \qquad (3.5)$$

■ **Algorithm 1 : The Spline Method for solving Falkner–Skan problems**

We use the following notations $b_0=\eta_\infty$, elfa=α, beta=β, Tk=t_k .

INPUTS:

Input (elfa, beta, b_0, Tk): the boundary conditions and constants.

Input (C_1,C_2,C_3), the initial approximation vector $C=(C_1,C_2,C_3)^T$.

Input ($(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3)$), the initial approximation vector $\tilde{C}=(\tilde{C}_1,\tilde{C}_2,\tilde{C}_3)^T$.

Input (N), number of subintervals N.

Input (M), maximum number of iterations M .

Tolerance Tol=0.1E-8; the parameters z1=0.5; z2= 2/3;

Step 1 Set h=(1)/(N);

ki=1;

Step 2 while (ki <= M) do {0} **Steps** 3-22.

Step 3
Begin {The initial conditions}
s0=0;
s1=0;
s2=tk;
s3=Sqr(b0)*(-elfa*s0*s2-beta*(1-s1*s1));
s4=Sqr(b0)*(-elfa*(s1*s2+s0*s3)+2*beta*s1*s2);
z0=0;
z1=0;
z2=1;
z3=Sqr(b0)*(-elfa*(z0*s2+s0*z2)+2*beta*z1*s1);
z4=Sqr(b0)*(-elfa*(s3*z0+s2*z1+s1*z2+s0*z3)+2*beta*(s2*z1+s1*z2));
Step 4 for r:=1 to N do {1} **steps** 5-19.
{Spline approximation (2.3a) for the nonlinear system (1.7)}
We set $S_f(C) \equiv S'''(C)+\alpha\eta_\infty^2 S(C) S''(C)+ \beta\eta_\infty^2(1-(S'(C)^2))=0$.
Step 5 k=1;
Step 6 while k <= M do {1} **steps** 7-11
Step 7 Calculate $S_f(C)$ and $J_f(C)$,
where $J_f(C)_{i,j} = \partial S_{f,i}(C)/\partial C_j$ for $1 \le i, j \le 3$.
Step 8 Solve the 3×3 linear system $J_f(C)Y = -S_f(C)$.
Step 9 Set C=C+Y.
Step 10 if ||Y|| < Tol then
spl[r]=S_r(C);
spl1[r]=S'_r(C);
spl2[r]=S''_r(C);
Step 11 Set k:=1+k;
{ end while 1}
{Spline approximation (2.6a) for the nonlinear system (1.14)}
We Set $S_z(\tilde{C}) \equiv Z'''(C)+\alpha\eta_\infty^2 Z(\tilde{C})S''(C)-2\beta\eta_\infty^2 Z'(\tilde{C})S'(C)+\alpha\eta_\infty^2 Z''(\tilde{C})S(C)=0$
Step 12 k=1;
steps 13 While (k<=m) do {2} **steps** 14-18.
Step 14 Calculate $S_z(\tilde{C})$ and $J_z(\tilde{C})$,
where $J_z(\tilde{C})_{i,j} = \partial S_{z,i}(\tilde{C})/\partial \tilde{C}_j$ for $1 \le i, j \le 3$.
Step 15 Solve the 3×3 linear system $J_z(\tilde{C})\bar{Y} = -S_z(\tilde{C})$.
Step 16 Set $\tilde{C} = \tilde{C} + \tilde{Y}$.
Step 17 if $\|\tilde{Y}\|$< Tol then
x:=z[i]*h;
sU[r]=$Z_r(\tilde{C})$;
sU1[r]=$Z'_r(\tilde{C})$;
sU2[r]=$Z''_r(\tilde{C})$;
Step 18 Set k:=k+1;
{ end while 2}
Step 19 Set { Renewing the initial conditions }
s0= S_r(C);

```
        s1= S`r(C);
        s2= S``r(C);
        s3=Sqr(b0)*(-elfa*s0* s2-beta*(1-s1*s1));
        s4=Sqr(b0)*(-elfa*(s1*s2+s0*s3)+2*beta*s1*s2);
        z0=Zr(C̃);
        z1=Z`r(C̃);
        z2=Z``r(C̃);
          z3=Sqr(b0)*(-elfa*(z0*s2+s0*z2)+2*beta*s1*z1);
        z4=Sqr(b0)*(-elfa*(s3*z0+s2*z1+s1*z2+s0*z3)+2*beta*(s2*z1+s1*z2));{end for 1}
Step 20   if abs(spl1[l,3]-1)<=tol then
 Step 21    for r:=1  to N do {2}
        Set  x=z[i]*h+(r-1)*h;
        x=x*b0;
        Spl[r]=Spl[r]*b0;
        Spl2[r]=Spl2[r]/b0;
        Output(x,Spl[r],Spl1[r],Spl2[r], Tk, Ki);
        { end for 2}
        Output ('Procedure is complete. ');
        STOP.
Step 22    tk:=tk-(spl1[N,3]-1)/su1[N,3];
        ki=ki+1;
        { end while 0}
Step 23          Output ('Maximum number of iterations exceeded');
        {Procedure completed unsuccessfully}
           END.
```

Results and Discussion:

In this section, we utilize Pascal programs to run the Algorithm1 of the presented Spline method for the numerical solution of the Falkner–Skan problems in numerous cases. We show the approximate spline solutions and their derivatives of special flows such as: Blasius, Pohlhausen, Homann and Hiemenz flows over various intervals. The approximate spline solutions of the wall shear stress of Falkner–Skan equation over various intervals are showed. We compute the local errors and the rate of convergence for present spline method.

In **Table 1**, we summarize spline approximations of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ for Blasius flow ($\alpha=1, \beta=0$), with *h*=0.1, $0 \le \eta \le 7.30$.

Comparisons of the values of the solution and its derivatives at $\eta=\eta_\infty$ corresponding to different $\beta \in [-0.1988, \infty[$ when $\alpha=1$ are listed in **Table 2**.

Table 3 shows comparisons of the wall shear stress $f''_w = f''(0)$ with *N*=24, for the present spline method. The results indicate that all values of f''_w are agreement with those reported by Beckett [2], El-Gindy et al. [4], El-Hawary [5] and Elbarbary **[**3**]**.

Fig.1 depicts the spline approximations of $df/d\eta$ for special flows: Blasius, Pohlhausen, Homann and Hiemenz flows over the interval $0 \le \eta \le 4$, with *N*=32, *h*=0.1875.

Fig. 2 plots spline approximations of $d2f/d\eta^2$ corresponding to $\alpha=1$ and different values for β, over the interval $0 \le \eta \le 8$, with *N*=32, *h*=0.1875.

Fig. 3 illustrates spline approximations of $df/d\eta$ corresponding to $\alpha = 1$ and different values for β, over the interval $0 \le \eta \le 6$, with N=32, h=0.1875.

Fig. 4 plots spline approximations of $f(\eta)$ corresponding to $\alpha = 1$ and different values for $\beta \in [-0.1988, \infty[$, over the interval $0 \le \eta \le 10$, with N=40, h=0.25.

Table 1: ***Spline approximations of the Falkner–Skan boundary-layer equation for Blasius flow*** **($\alpha = 1, \beta = 0$)**

η	$f(\eta)$		$f'(\eta)$		$f''(\eta)$	
	Kuo[7]	Present Method	Kuo[7]	Present Method	Kuo[7]	Present Method
0.00	0.000000	0.000000000	0.000000	0.000000000	0.469600	0.469599988
0.10	0. 002348	0.002347982	0.046959	0.046959080	0.469563	0.469563236
0.20	0.009391	0.009391412	0.093905	0.093905299	0.469306	0.469306057
0.30	0.021128	0.021127536	0.140806	0.140805619	0.468609	0.468608782
0.40	0.037549	0.037549200	0.187605	0.187605139	0.467254	0.467254198
0.50	0.058643	0.058642681	0.234228	0.234227473	0.465030	0.465030358
0.60	0.084386	0.084385571	0.280575	0.280575459	0.461734	0.461734432
0.70	0.114745	0.114744754	0.326532	0.326532304	0.457178	0.457177485
0.80	0.149674	0.149674519	0.371963	0.371963244	0.451190	0.451190021
0.90	0.189115	0.189114871	0.416718	0.416717789	0.443628	0.443628017
1.00	0.232990	0.232990096	0.460633	0.460632577	0.434379	0.434379146
150	0.515031	0.515031531	0.661474	0.661473834	0.361804	0.361804520
2.00	0.886797	0.886796828	0.816695	0.816694624	0.255669	0.255669173
2.40	1.231528	1.231527622	0.901065	0.901065445	0.167560	0.167560358
3.00	1.795568	1.795567915	0.969055	0.969054607	0.067710	0.067710344
3.40	2.187467	2.187467253	0.987970	0.987970463	0.030535	0.030535263
3.80	2.584499	2.584498788	0.995944	0.995944282	0.030535	0.011758707
4.00	2.783886	2.783886463	0.997770	0.997770098	0.006874	0.006874106
4.40	3.183383	3.183382702	0.999397	0.999396613	0.002084	0.002084083
4.80	3.583254	3.583254069	0.999859	0.999859402	0.000538	0.000538487
5.00	3.783235	3.783234525	0.999936	0.999935865	0.000258	0.000257782
7.30	--------	6.083219379	---------	1.000000000	---------	0.000000003

Table 2: ***Comparisons of the values of the solution and its derivatives at*** $\eta = \eta_\infty$ ***corresponding to different*** $\beta \in [-0.1988, \infty[$ ***when*** $\alpha = 1$.

	Iterative Method [13]			Present Method , N=40		
β	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
2	4.232790	1.000000	0.000001	6.302566325	1.0000000019	0.0000000011
1	4.560670	1.000000	0.000001	7.352099526	1.0000000000	0.0000000000
0.5	4.982643	1.000000	0.000000	7.195451385	1.0000000000	0.0000000000
0.0	5.247298	1.000000	0.000000	6.783219378	1.0000000000	0.0000000000
-0.1	5.313184	1.000000	0.000000	6.557303204	1.0000000000	0.0000000002
-0.15	5.343293	1.000000	0.000000	6.353030458	1.0000000000	0.0000000010
-0.18	5.375395	1.000000	0.000000	6.128424599	1.0000000009	0.0000000045
-0.1988	5.378092	1.000000	0.000000	5.667023693	1.0000000000	0.0000000730

Table 3: ***Comparison of the wall shear stress*** f''_w **.**

η_∞	α	β	Beckett [2]	El-Gindy et al. [4]	El-Hawary [5]	Elbarbary [3]	Present Method
2	1	15	4.4923	4.4905	4.4916430	4.49148688	4.491486878
2	1	10	3.6756	3.6746	3.6752130	3.67523431	3.675234306
3.1	1	2	1.6874	1.7741	1.6872256	1.68722586	1.687225863
3.7	1	0.5	0.92778	0.694	0.9278054	0.92780539	0.927805388
4.4	1	0.3	-----	0.5332	0.7747827	0.77478274	0.774782741
6.9	1	0.0	0.4697	-----	0.4696000	0.46960012	0.469599992

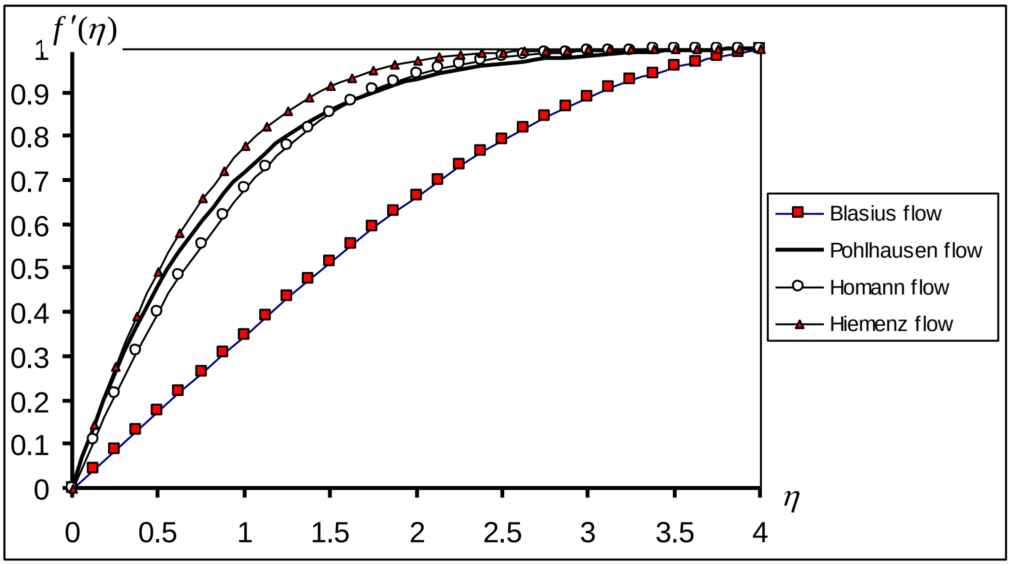

Fig.1. Spline approximations of $df/d\eta$ **for special flows: Blasius, Pohlhausen, Homann and Hiemenz flows over the interval** $0 \le \eta \le 4$**, with N=32, h=0.1875 .**

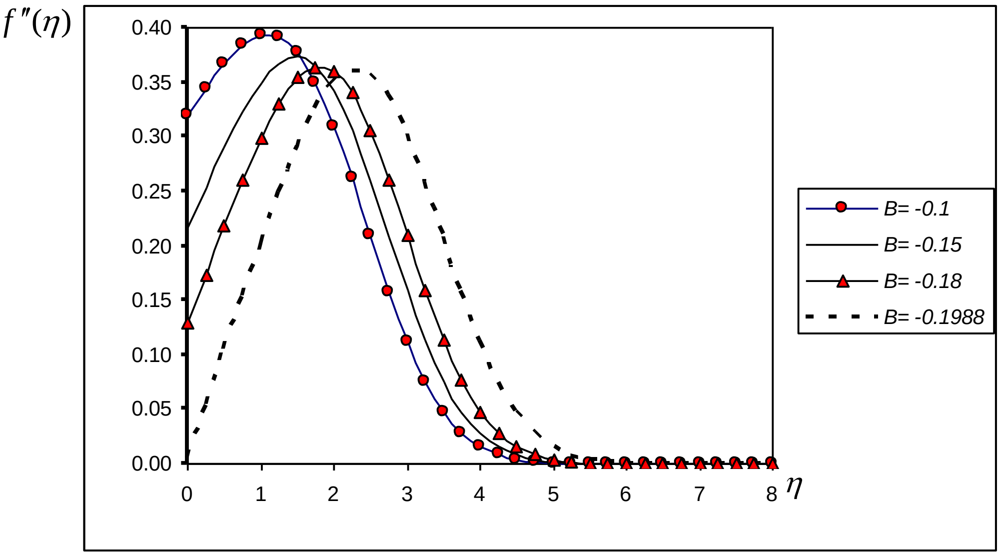

Fig. 2. Spline approximations of $d2f/d\eta^2$ **corresponding to** $\alpha = 1$ **and different values for** β**, with N=32, h=0.1875.**

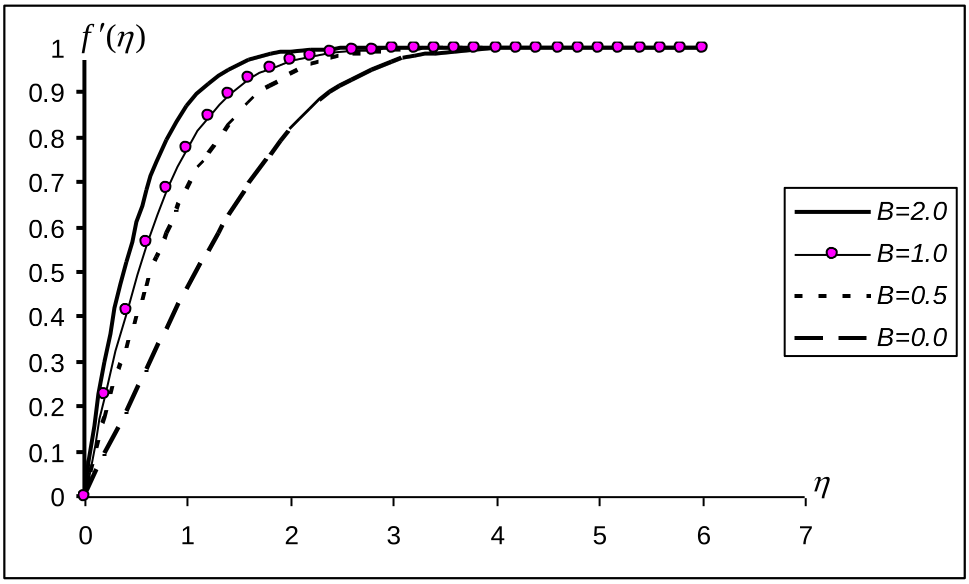

Fig. 3. spline approximations of $df/d\eta$ corresponding to $\alpha = 1$ and different values for β, with N=32, h=0.1875.

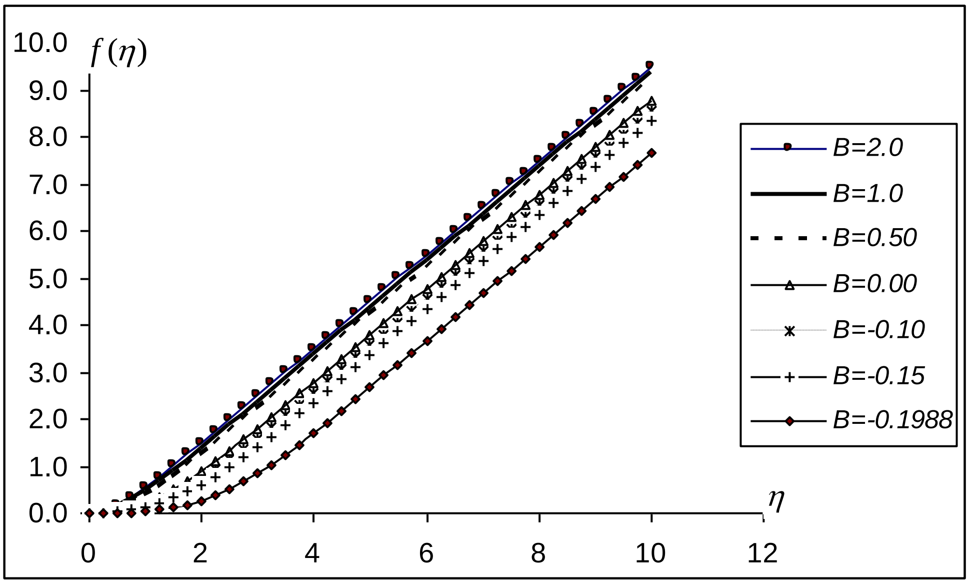

Fig. 4. ***Spline approximations of $f(\eta)$ corresponding to $\alpha = 1$ and different values for β, with N*=40, *h*=0.25.**

Error Estimates: We solve the Falkner–Skan problem (case 1: Blasius flow) for $h = \eta_\infty / N$, with prescribed error tolerance $\varepsilon = 10^{-8}$. The nodal difference error E_k^N, is defined by:

$$E_k^N = |S_k^N - S_{2k}^{2N}|, \quad k=1,\ldots,N$$

where S_k^N is the spline approximation at η_k by the present spline method. The experimental nodal rate of convergence is given by:

$$Rate = Log_2(E^N / E^{2N})$$

Table 4 shows the local errors at the *k*th step for present spline method, with $\eta_\infty = 10$, *N*=40. By solving the Falkner–Skan problem for $\eta_\infty = 10$, *N*=20, 40, 80, the order of convergence for the proposed spline method is computed in **Table 5**.

Table 4: The local errors for Falkner–Skan problem by presented spline method when, N=40, h=0.25

k	E_k^N	$E_k'^N$	$E_k''^N$	$E_k'''^N$
1	1.5049300E-11	2.5670489E-10	5.1841198E-11	6.30250018E-12
2	9.2256897E-11	4.8180482E-10	2.0463686E-11	4.172310066-11
4	3.6243369E-10	5.0522425E-10	2.4101609E-11	2.13617499E-10
6	4.9749360E-10	2.5102054E-10	7.5124273E-11	5.6707000E-10
8	2.9558578E-10	5.7389116E-10	5.661603E-10	5.77756509E-10
10	1.5825208E-10	2.6284397E-10	9.7884370E-10	1.271246219E-9
15	5.2386895E-10	1.6916601E-10	6.0254020E-10	1.534260698E-9
20	4.0017767E-10	1.90993887E-11	5.9777899E-11	2.26051999E-10
25	4.2928150E-10	6.3664629E-12	2.852200E-12	1.43556899E-10
30	4.3655746E-10	9.0949470E-13	2.4640000E-13	1.548499999E-12
35	4.3655746E-10	9.0949470E-13	2.9999999E-16	1.89999999E-16
40	4.3655746E-10	9.0949470E-13	0.0	0.0

Table 5: The rate of convergence for presented spline method , with N=20.

k	$E_k^N = \lvert S_k^N - S_{2k}^{2N} \rvert$	$E_k^{2N} = \lvert S_{2k}^{2N} - S_{4k}^{4N} \rvert$	Rate of convergence
1	3.503089305 E-9	9.225689657 E-11	5.24683
2	1.886564863 E-8	3.624336942 E-10	5.70190
3	2.947035682 E-8	4.974936018 E-10	5.88844
4	1.986154529 E-8	2.955857781 E-10	6.07026
5	8.756614988 E-9	1.582520781 E-10	5.79008
6	1.832086127 E-8	3.583409125 E-10	5.67601
7	3.078821464 E-8	5.275069270 E-10	5.86704
8	2.980596037 E-8	4.874891601 E-10	5.93409
9	2.453816705 E-8	4.147295840 E-10	5.88671
10	2.290107659 E-8	4.001776687 E-10	5.83863
11	2.349406713 E-8	4.220055416 E-10	5.79889
12	2.409069566 E-8	4.292814992 E-10	5.81041
13	2.430897438 E-8	4.292814992 E-10	5.82342
14	2.435263013 E-8	4.365574568 E-10	5.80176
15	2.435990609 E-8	4.365574568 E-10	5.80219
16	2.4359906092 E-8	4.365574568 E-10	5.80219
17	2.4359906092 E-8	4.365574568 E-10	5.80219
18	2.4359906092 E-8	4.365574568 E-10	5.80219
19	2.4359906092 E-8	4.365574568 E-10	5.80219
20	2.4359906092 E-8	4.365574568 E-10	5.80219

In Tables 1,2, we compare our results of the spline method to the results obtained [2-5,7,13]. In Table 1, we find that the spline solution $f(\eta)$ and its derivatives of the Blasius flow problem are better than those obtained by the KUO method [7]. Table 2 shows that the spline solution to $f'(\eta_\infty)$ is at least of eight-decimal-place accuracy, and the spline solution to $f''(\eta_\infty)$ is at least of seven-decimal-place accuracy, while the results of iterative method by Zhang [13] is at least of six and five decimal-place accuracy, respectively. Also, comparisons of the wall shear stress f''_w between our results and other results [2-5] are summarized in Tables 3. The Figs. 1-4, illustrate that the suggested technique is quite reliable. Tables 4-5, come into view that the maximum local error of spline method is 1.6×10^{-9} for each step h=0.25, moreover, the rate of convergence is at least five.

Conclusion:

We have presented an iterative method based on polynomial splines for solving the Falkner–Skan problems over semi-infinite intervals. This proposed method enables us to approximate the solution and its derivatives of the problem at every point of the range of integration. The collocation points applied into the polynomial spline function to be satisfied by parameters $z_1 = 1/2,\ z_2 = 2/3, z_3 = 1$ improve the accuracy of the scheme, which are evident form the numerical results given in Tables 1-4. The results obtained are very encouraging and the spline method performs better than the previous methods in [2-5,7,13] . Finally, since the applicability and efficiency of the presented spline method, we recommend its using for solving Falkner–Skan problems and all nonlinear third-order boundary-value problems.

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