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Quintic C^2- Spline Collocation Methods for Solving Initial Value Problems in Higher Index Differential-Algebraic Equations

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□ ABSTRACT □

In this paper, a class of quintic C^2- spline collocation methods when applied to differential-algebraic systems with index greater than or equal one is presented.

These methods do not in general attain the same order of accuracy for higher index differential-algebraic systems as they do for index-1 systems. We prove that the proposed methods if applied to index-1 systems are stable and consistent of order five, while they are stable and consistent of order four for index greater than one. Necessary and sufficient conditions on parameters $c_1, c_2 \in]0,1[$ of the methods are derived to ensure that the methods applied to index-v systems are strictly stable. By giving four numerical examples and comparing with other methods, the applicability and efficiency of the methods are shown.

Key Words: Differential-Algebraic Equations, Spline Collocation Methods, Higher-index, Stability, Consistency, Convergence, Strict Stability.

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طرائق شرائحية تجميعية في C^2 لحل مسائل القيمة الابتدائية في المعادلات التفاضلية الجبرية ذات الدليل العالي

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□ ملخّص □

تم في هذا البحث تقديم طرائق شرائحية تجميعية من الدرجة الخامسة في C^2 عند تطبيقها للحل العددي للمعادلات التفاضلية الجبرية ذات دليل أكبر أو يساوي الواحد. تبين الدراسة أن الطرائق لا تملك في الحالة العامة نفس الرتبة من الدقة عند تطبيقها لمعادلات تفاضلية جبرية دليلها أكبر من الواحد؛ فالطرائق تكون مستقرة ومتناسقة ومتقاربة من الرتبة الخامسة عند تطبيقها لأنظمة دليلها يساوي الواحد، بينما هذا الاستقرار والتناسق والتقارب يكون من الرتبة الرابعة إذا كان دليل هذه الأنظمة أكبر من الواحد. نحدد بعض الشروط الضرورية والكافية على وسيطيْ الطرائق c_1, c_2 في المجال $]0,1[$ لضمان الاستقرار الأكيد للطرائق المقدمة.

وقد تم اختبار فعالية الطرائق المقدمة بحل أربع مسائل ذات أدلة مختلفة حيث تشير النَتائِج العددية إلى فعالية وكفاءة هذه الطرائق مقارنة مع بعض الطرائق الأخرى.

الكلمات المفتاحية: معادلات تفاضلية جبرية، طرائق شرائحية تجميعية، دليل عال، الاستقرار، التناسق، التقارب، الاستقرار الأكيد.

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Introduction:

Differential algebraic equations (DAEs) arise in many instances when using mathematical modeling techniques for describing phenomena in science, engineering, economics, etc. In the last three decades, the use of differential algebraic equations has become standard modeling practice in many applications, such as constrained mechanics and chemical process simulations. In most cases, the model is too complex to allow one to find an exact solution or even an approximate solution by hand: an efficient, reliable computer simulation is required. It is well known that DAEs can be difficult to solve when they have a higher index, i.e., an index greater than one (cf. [**3**]). Higher-index DAEs are ill posed in a certain sense, especially when the index is greater than two [**1**], and a straightforward discretization generally does not work well. Some numerical methods have been developed, using Runge–Kutta, BDF and regularization methods [**2**,**3**,**4**,**9**,**11**,**14**]. Differential transform method introduced by Liu in [**10**], who solved linear problems for index only two and three. A multi-resolution collocation method with specially designed spline wavelet is presented to numerically solve a system of nonlinear differential-algebraic equations of 1-index in [**5**]. In [**7**], Ayaz gave two numerical examples to illustrate the efficiency of the method, but the two examples are all index-1 DAEs. Linear differential-algebraic equations with properly stated leading term: Regular points by März and Riaza in [**13**].

1. Importance and Aim of This Research

The main purpose of the paper is to develop a class of Quintic Spline Collocation Methods (QSCMs) when applied to differential-algebraic systems with index greater than or equal one. It is well known that DAEs can be very difficult to solve when they have a higher index, i.e., an index greater than one.

2. Methodology

First, in **theoretical part**: we discuss the error analysis and order of convergence of spline approximations methods applied to solvable linear constant coefficient DAEs

$$\mathbf{A}\, y' + \mathbf{B}\, y = g(t)\,, \tag{1.1}$$

of arbitrary index-v, where A and B are square constant matrices and $g(t)$ is a smooth function. After that, we study the strict stability properties of the spline approximations applied to nonlinear systems of DAEs of the form,

$$F(\,t\,,\; y(t),\; y'(t))=0\,, \tag{1.2}$$

where the initial values of $y(t_0)$ are given and F is linear in y'.

Denote by $t_i = a + i\,h,\ i = 0(1)\mathrm{N}$, the grid points of the uniform partition of [a,b] into subintervals $\mathrm{I}_i = [t_{i-1}, t_i],\ i = 1(1)\mathrm{N}$. A fifth C^2-spline functions $S(x)$ can be represented on each I_i [**12**], by

$$\begin{aligned} S(t) = \bar{T}^3[(6T^2+3T+1)S_{i-1}^{(0)} + (3T^2+T)S_{i-1}^{(1)} + (\tfrac{1}{2}T^2)S_{i-1}^{(2)}] \\ + T^3[(6\bar{T}^2+3\bar{T}+1)S_i^{(0)} - (3\bar{T}^2+\bar{T})S_i^{(1)} + (\tfrac{1}{2}\bar{T}^2)S_i^{(2)}] \end{aligned}\,, \tag{1.3}$$

where $T = \dfrac{(t-t_{i-1})}{h},\quad \bar{T} = 1 - T;\quad T,\ \bar{T} \in [0,1]$, and

$$S_i^{(0)} = S(t_i),\ S_i^{(1)} = hS'(t_i),\ S_i^{(2)} = h^2 S''(t_i),\quad i = 0(1)\mathrm{N}\,. \tag{1.4}$$

Differentiating (1.3), we have

$$h\,S'(t) = \bar{T}^2[-30T^2\,S_{i-1}^{(0)} + (1+2T-15T^2)S_{i-1}^{(1)} + (T-\tfrac{5}{2}T^2)S_{i-1}^{(2)}] - T^2[-30\,\bar{T}^2\,S_i^{(0)} - (1+2\,\bar{T}-15\,\bar{T}^2)S_i^{(1)} + (\bar{T}-\tfrac{5}{2}\bar{T}^2)\,S_i^{(2)}] \quad . \qquad (1.5)$$

The spline approximations use three collocation points $t_{i-1+c_j} = t_{i-1} + c_j h,\ j = 1(1)3$, in each subinterval I_i , $i = 1(1)N$, with $c_1, c_2 \in]0, 1[$, $c_3=1$, $c_1 \neq c_2$ be fixed and $h = (b-a)/N$ is the constant stepsize.

We formally apply the spline approximations (1.3)-(1.5) to the DAE (1.2) to obtain the system

$$\begin{aligned} &F[t_{i-1+c_1},\ S(t_{i-1+c_1}),\ S'(t_{i-1+c_1})]) = 0\ , \\ &F[t_{i-1+c_2},\ S(t_{i-1+c_2}),\ S'(t_{i-1+c_2})] = 0\ , \quad i = 1(1)N\ , \qquad (1.6) \\ &F[t_i,\ S(t_i),\ S'(t_i)] = 0\ , \end{aligned}$$

with initial-values:

$$S(t_0) = S_0^{(0)}, S'(t_0) = h^{-1}S_0^{(1)}, S''(t_0) = h^{-2}S_0^{(2)} \quad . \qquad (1.7)$$

Practical part: numerical experiments are presented that illustrate the theoretical results. We have accomplished the computations by using programs *Mathematica* Version 5.0.0.0 and Turbo Pascal in double precision.

3. Paper Outline

The paper is organized as follows: In Section 2, the case of linear constant coefficient index-v systems is studied. It shows that the Quintic C^2- spline collocation methods when applied index-1 systems are stable (Corollary1), consistent of order **five** (Theorem1), and convergent of order **five** (Corollary2). After that, we generalize the *QSCM*s when applied to differential-algebraic systems with index greater than one. It turns out that proposed *QSCM*s are stable (Corollary3) and consistent of order **four** for all $v \geq 2$. In Section 3 the *QSCM*s are shown to be strictly stable if applied to index-v DAEs for all $0.949 \leq c_1 < c_2 < 1$ (Theorem 2). In Section 4, we present numerical experiments to test the efficiency of the *QSCM*s when applied to differential-algebraic systems for both linear and nonlinear problems.

Linear constant coefficient systems:

In this section, we consider linear constant coefficient systems of arbitrary index-ν. We derive conditions that are sufficient to ensure the order, stability, consistency and convergence of the *QSCM*s when applied to these systems.

Consider the linear constant coefficient DAE (1.1)

$$\mathbf{A}\,y' + \mathbf{B}\,y = g(t) \qquad (2.1)$$

of index-v. We assume this system is solvable, so that there exist nonsingular matrices P and Q such that,

$$P\mathbf{A}Q = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad P\mathbf{B}Q = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \qquad (2.2)$$

where $\boldsymbol{I}$ is an identity matrix and M is a block diagonal matrix, M=diag (M_1, M_2, …, M_L) composed of blocks of the form

$$M_k = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}. \qquad (2.3)$$

Applying the spline approximations (1.3)-(1.5) to (2.1), we get

$$\mathbf{A}S'(t_{i-1+c_j}) + \mathbf{B}S(t_{i-1+c_j}) = g(t_{i-1+c_j}),\ j=(1)3\ ,\quad i=1(1)\mathrm{N}, \qquad (2.4a)$$

with initial-values:

$$S(t_0)=S_0^{(0)}, S'(t_0)=h^{-1}S_0^{(1)}, S''(t_0)=h^{-2}S_0^{(2)}\ . \qquad (2.4b)$$

Let $\tilde{S}(t_{i-1+c_j}) = Q^{-1}S(t_{i-1+c_j})$, $\tilde{S}'(t_{i-1+c_j}) = Q^{-1}S'(t_{i-1+c_j})$, $\tilde{g}(t_{i-1+c_j}) = P\,g(t_{i-1+c_j})$, and premultiplying by P, we can rewrite (2.4a) as

$$(P\mathbf{A}Q)\tilde{S}'(t_{i-1+c_j}) + (P\mathbf{B}Q)\tilde{S}(t_{i-1+c_j}) = \tilde{g}(t_{i-1+c_j}),\ j=1(1)3.$$

In this form, the differential and algebraic parts of the system are completely decoupled from each other. Thus, it is sufficient to study the differential and algebraic parts separately to get an understanding of the general linear constant-coefficient DAE.

Consider then a canonical algebraic subsystem of index-v

$$M\,y' + y = g(t)\ , \qquad (2.5)$$

where M is a $v\times v$ matrix of the form (2.3), $g(t)=(g_1(t),\cdots,g_v(t))^T$, and $y(t)=(y_1(t),\cdots,\ y_v(t))^T$. The solution to (2.5) is given by

$$\begin{aligned} y_1(t) &= g_1(t) \\ y_2(t) &= g_2(t) - y_1'(t) \\ &\vdots \\ y_v(t) &= g_v(t) + (-1)^{v-1}y'_{v-1}(t) \end{aligned}$$

Applying the approximations (1.3)-(1.5) into (2.5), we obtain

$$M\,S'(t_{i-1+c_j}) + S(t_{i-1+c_j}) = g(t_{i-1+c_j})\ ,\quad j=1(1)3\ ,\quad i=1(1)\mathrm{N}. \qquad (2.6)$$

Let $S=(S_1,\ldots,S_v)^T, S'=(S_1',\ldots,S_v')^T$, where S_j' denotes the derivative corresponding to the j th component of the solution vector.

1. The Methods QSCMs for Index-1 Problem.

We assume that the methods are applied to index-1 systems, then (2.6) reduces to a set of algebraic equations of the form

$$S_1(t_{i-1+c_j}) = g_1(t_{i-1+c_j})\ ,\quad j=1(1)3\ ,\quad i=1(1)\mathrm{N}, \qquad (2.7a)$$

with initial values:

$$S_1(t_0)=S_{1,0}^{(0)},\ \ S_1'(t_0)=h^{-1}S_{1,0}^{(1)},\ \ S_1''(t_0)=h^{-2}S_{1,0}^{(2)}\ . \qquad (2.7b)$$

By using the approximation (1.3) into (2.7a) , i.e., taking $S_1 \equiv S$, we obtain

$$\begin{aligned} &\bar{c}_j^3[(6c_j^2+3c_j+1)S_{1,\,i-1}^{(0)} + (3c_j^2+c_j)S_{1,\,i-1}^{(1)} + (\tfrac{1}{2}\,c_j^2)S_{1,\,i-1}^{(2)}] + \\ &c_j^3[(6\bar{c}_j^2+3\bar{c}_j+1)S_{1,\,i}^{(0)} - (3\bar{c}_j^2+\bar{c}_j)S_{1,\,i}^{(1)} + (\tfrac{1}{2}\,\bar{c}_j^2)S_{1,\,i}^{(2)}] = g_1(t_{i-1+c_j})\ ,\ j=1(1)2 \end{aligned} \qquad (2.8a)$$

$$S_{1,i}^{(0)} = g_1(t_i)\ . \qquad (2.8b)$$

where $\bar{c}_j = 1 - c_j,\ j=1(1)3.$

Substituting $S_{1,\,i}^{(0)} = g_1(t_i),\ S_{1,\,i-1}^{(0)} = g_1(t_{i-1})$ into (2.8a), we have

$$c_j^3[-(3\bar{c}_j^2+\bar{c}_j)S_{1,i}^{(1)}+(\tfrac{1}{2}\bar{c}_j^2)S_{1,i}^{(2)}]=-\bar{c}_j^3[(3c_j^2+c_j)S_{1,i-1}^{(1)}+(\tfrac{1}{2}c_j^2)S_{1,i-1}^{(2)}]-$$
$$\bar{c}_j^3(6c_j^2+3c_j+1)\,g_1(t_{i-1})+g_1(t_{i-1+c_j})-c_j^3(6\bar{c}_j^{\,2}+3\bar{c}_j+1)\,g_1(t_i)\quad , \qquad (2.9)$$
$$j=1(1)2,$$

which are equivalent to the following recurrence formula:

$$\mathrm{A}_1\underline{S}_{1,i}=\mathrm{B}_1\underline{S}_{1,i-1}+\mathrm{D}_1\underline{g}_{1,i}\quad ,i=1(1)\mathrm{N}, \qquad (2.10)$$

where

$$\mathrm{A}_1=\begin{bmatrix}-c_1^3(3\,\bar{c}_1^2+c_1') & \tfrac{1}{2}c_1^3\,\bar{c}_1^2\\ -c_2^3(3\bar{c}_2^2+\bar{c}_2) & \tfrac{1}{2}c_2^3\,\bar{c}_2^2\end{bmatrix},\quad \mathrm{B}_1=\begin{bmatrix}-\bar{c}_1^3(3c_1^2+c_1) & -\tfrac{1}{2}\,\bar{c}_1^3\,c_1^2\\ -\bar{c}_2^3(3c_2^2+c_2) & -\tfrac{1}{2}\,\bar{c}_2^3\,c_2^2\end{bmatrix},$$

$$\mathrm{D}_1=\begin{bmatrix}-\bar{c}_1^3(6c_1^2+3c_1+1) & 1 & 0 & -c_1^3(6\bar{c}_1^2+3\bar{c}_1+1)\\ -\bar{c}_2^3(6c_2^2+3c_2+1) & 0 & 1 & -c_2^3(6\bar{c}_2^2+3\bar{c}_2+1)\end{bmatrix},$$

and

$$\underline{S}_{1,i}=\begin{bmatrix}S_{1,i}^{(1)}\\ S_{1,i}^{(2)}\end{bmatrix},\quad \underline{S}_{1,i-1}=\begin{bmatrix}S_{1,i-1}^{(1)}\\ S_{1,i-1}^{(2)}\end{bmatrix},\quad \underline{g}_{1,i}=\begin{bmatrix}g_{1,i-1}\\ g_{1,i-1+c_1}\\ g_{1,i-1+c_2}\\ g_{1,i}\end{bmatrix}.$$

Multiplying (2.10) by A_1^{-1}, we get

$$\underline{S}_{1,i}=\tilde{\mathrm{A}}_1\underline{S}_{1,i-1}+\mathrm{A}_1^{-1}\mathrm{D}_1\,\underline{g}_{1,i}\quad ,i=1(1)\mathrm{N}, \qquad (2.11a)$$

where

$$\tilde{\mathrm{A}}_1=\mathrm{A}_1^{-1}\mathrm{B}_1=$$

$$\begin{bmatrix}-\dfrac{\bar{c}_1\bar{c}_2(c_1+c_2+2c_1c_2)}{c_1^2c_2^2} & \dfrac{-\bar{c}_1\,\bar{c}_2}{2c_1\,c_2}\\[2ex] \dfrac{-2[4c_1+4c_2+c_2c_1-3(c_2^2+c_1^2+c_2c_1^2+c_1\bar{c}_1c_2^2)]}{c_1^2c_2^2} & \dfrac{3c_1+3c_2-2c_1c_2-4}{c_1c_2}\end{bmatrix} \qquad (2.11b)$$

If $0<c_1<c_2<1$, then $\tilde{\mathrm{A}}_1=\mathrm{A}_1^{-1}\mathrm{B}_1$ exists because

$$|A_1|=\tfrac{1}{2}(1-c_1)(1-c_2)(c_2-c_1)c_1^3c_2^3\neq 0,\text{and}\quad |\tilde{\mathrm{A}}_1|=\frac{(1-c_1)^2(1-c_2)^2}{c_1^2c_2^2}\neq 0.$$

The QSCMs when applied to index-1 system (2.7a)-(2.7b) will be analyzed in the form (2.11a) .

Definition 1 [**6**]: The QSCMs (2.11) are called stable if $\|(\tilde{A}_1)^n\|\le k=const$ for all $n\ge 1$, where $k=\max\limits_{1\le i\le 3}\sum_{j=1}^{3}|a_{i,j}^n|$, $\tilde{A}_1^{\,n}=[a_{i,j}^n]$, and $\tilde{A}_1$ is the matrix (2.11b).

Corollary 1. The QSCMs applied to index-1 systems are stable if eigenvalues of the matrix $\tilde{\mathrm{A}}_1$ satisfy

$$|\mu_1|,\ |\mu_2|\le 1, \qquad (2.13)$$

for $c_1, c_2 \in]0,1[$, and $c_1 \neq c_2$.

Proof. The spline methods (2.7a)-(2.7b) are stable for index-1 systems if $\|\tilde{A}_1^n\|_\infty$ is uniformly bounded for all $n \geq 1$, according to the **definition 1** of stability. Moreover if $|\mu_1|, |\mu_2| \leq 1$ are satisfied, then $\|\tilde{A}_1^n\|_\infty \leq k$, $k = \max_{1 \leq i \leq 2} \sum_{j=1}^{2} |a_{ij}^n|$, where $\tilde{A}_1^n = (a_{ij}^n)$, and also we get $k \to 0$ as $n \to \infty$. Therefore the matrix $\tilde{A}_1$ has two different eigenvalues, for some c_1, c_2, the computations of eigenvalues are given in Table1. In Table2, we show than $\|\tilde{A}_1^n\|_\infty \leq 5.4580$, $\forall\ n \geq 1$ for example the method ($c_1 = 0.65, t_2 = 0.999$)

Table 1: The methods (2.7a)-(2.7b) are stable for some values of c_1, c_2

The method (c_1, c_2)		The eigenvalues	
c_1=0.50	c_2=0.9998	μ_1=-3.994E-8	μ_2=-1.00
c_1=0.53	c_2=0.9940	μ_1=-3.047E-5	μ_2=-0.9404
c_1=0.57	c_2=0.9998	μ_1=-3.013E-8	μ_2=-0.755907
c_1=0.60	c_2=0.99	μ_1=-6.169E-5	μ_2=-0.735135
c_1=0.65	c_2=0.999	μ_1=-5.339E-7	μ_2=-0.544065
c_1=0.75	c_2=0.86	μ_1=-3.086E-3	μ_2=-0.954068
c_1=0.80	c_2=0.81	μ_1=-3.514E-3	μ_2=-0.978606
c_1=0.8028	c_2=0.80281	μ_1=-3.641E-3	μ_2=-0.999948
$\forall c_1, c_2 \in]0.8028,1[$, and $c_1 \neq c_2$ for example, c_1=0.81, c_2=0.95		μ_1=-0.000392	μ_2=-0.389119

Table 2: The norm $\|\tilde{A}_1^n\|_\infty$ is uniformly bounded for all $n \geq 1$, for $c_1 = 0.65, t_2 = 0.999$

n	1	2	5	10	20	$n \to \infty$
$\|\tilde{A}_1^n\|_\infty$	k=5.4580	k=2.9695	k= 0.4783	k=0.0229	k= 5.1807E-5	$k \to \infty$

Definition 2 [6]: The Quintic C^2- spline collocation method is said to be consistent of order p if $\max_{0 \leq i \leq n} \|d_i\| = O(h^p)$, where d_i is local discretization error of (2.11a) at t_i.

To find the local error, let $y_1(t_{i-1}) = g_1(t_{i-1})$.

Theorem 1. Let $y_1 \in C^6[a,b]$, then the methods (2.7a)-(2.7b) applied to index-1 systems are consistent and are of order **five**, for all c_1, c_2 shown in Table 1.

Proof. The local discretization error of (2.7a)-(2.7b) at t_i is defined to be

$$\underline{d}_{i,1}=\begin{bmatrix} h\,y_1'(t_i) \\ h^2 y_1''(t_i)\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{B}_1\begin{bmatrix} h\,y_1'(t_{i-1}) \\ h^2 y_1''(t_{i-1})\end{bmatrix}-\mathrm{A}_1^{-1}\mathrm{D}_1\begin{bmatrix} y_1(t_{i-1}) \\ y_1(t_{i-1+c_1}) \\ y_1(t_{i-1+c_2}) \\ y_1(t_i)\end{bmatrix},\ i=1(1)\mathrm{N}, \qquad (2.14)$$

where $y_1(t)$ is the exact solution. Now using Taylor's expansion

$$y_1(t)=\sum_{r=0}^{5}\frac{(t-t_{i-1})^r}{r!}y_1^{(r)}(t_{i-1})+\frac{(t-t_{i-1})^6}{6!}y_1^{(6)}(t_{i-1}) \quad ,t\in[\mathrm{t}_{i-1},t_i],$$

and applying to (2.14), we get

$$\underline{d}_{i,1}=\begin{bmatrix} -\frac{1}{720}(\bar{c}_1\,\bar{c}_2) \\ \frac{1}{360}(4c_1+4c_2-3c_1c_2-5)\end{bmatrix} y_1^{(6)}(t_{i-1})\,h^6\equiv O(h^6),\ i=1(1)\mathrm{N}.$$

Observing that $\underline{d}_{i,1}=0$ for Taylor polynomials of degree ≤ 5, in these cases the methods are exact. We deduce, according to Definitions 2, that the methods are thus consistent and are of order five for all c_1, c_2 in Table 1. □

<u>Corollary 2</u>: Let $y_1\in C^5[a,b]$ be Lipschitz continuous, then the approximation $S_1(t)$ converges to the solution $y_1(x)$ as $h\to 0$ whenever (2.13) is fulfilled and

$$\lim_{h\to 0} h^{-i}S_{1,0}^{(j)}=y_1^{(j)}(t_0),\ \ j=0,1,2.$$

Furthermore, the convergence order is five, i.e., we have

$$|\,y_1(t_i)-S_1^{(0)}(t_i)\,|\le C_0h^5,\ \ i=1(1)\mathrm{N}\ , \qquad (2.15a)$$

$$|\,y_1^{(r)}(t_i)-\frac{1}{h^r}S_1^{(r)}(t_i)\,|\le C_rh^{5-r},\ r=1,2,\quad i=1(1)\mathrm{N}\ , \qquad (2.15b)$$

whenever the initial-values (2.4b) satisfy (2.15). In addition, the following global error estimate holds true:

$$\left|y_1(t)-S_1(t)\right|=\frac{T^3\,\bar{T}^3}{720}\,y_1^{(6)}(t_{i-1})h^6\equiv O(h^6),\ t\in[t_{i-1},t_i].$$

2. The Methods QSCMs for Index-2 Systems.

Now, applying QSCMs to **index-2**, then (2.6) reduces to a set of algebraic equations as follows:

$$S_1(t_{i-1+c_j})=g_1(t_{i-1+c_j}), \qquad (2.16a)$$

$$S_2(t_{i-1+c_j})=g_2(t_{i-1+c_j})-g_1'(t_{i-1+c_j})\ \ ,\ j=1(1)3\ ,\ i=1(1)\mathrm{N}. \qquad (2.16b)$$

Also, applying the approximations (1.3)-(1.4) into (2.16b), i.e., taking $S_2\equiv S$ we have

$$c_j^3[(6\bar{c}_j^2+3\bar{c}_j+1)S_{2,i}^{(0)}-(3\bar{c}_j^2+\bar{c}_j)S_{2,i}^{(1)}+(\tfrac{1}{2}\,\bar{c}_j^2)S_{2,i}^{(2)}]+$$
$$\bar{c}_j^3[(6c_j^2+3c_j+1)S_{2,i-1}^{(0)}+(3c_j^2+c_j)S_{2,i-1}^{(1)}+(\tfrac{1}{2}c_j^2)S_{2,i-1}^{(2)}]= \qquad (2.17a)$$
$$g_2(t_{i-1+c_j})-g_1'(t_{i-1+c_j}) \qquad ,\ j=1,2$$

$$S_{2,i}^{(0)}=g_2(t_i)-g_1'(t_i)\ \ . \qquad (2.17b)$$

Substituting $S_{2,i}^{(0)}=g_2(t_i)-g_1'(t_i)\,, S_{2,i-1}^{(0)}=g_2(t_{i-1})-g_1'(t_{i-1})$, into (2.17a), we get

$$-c_j^3(3\,\bar{c}_j^2+\bar{c}_j)S_{2,\,i}^{(1)}+\tfrac{1}{2}c_j^3\,\bar{c}_j^2S_{2,\,i}^{(2)}=-\bar{c}_j^3(3c_j^2+c_j)S_{2,\,i-1}^{(1)}-\tfrac{1}{2}\bar{c}_j^3\,c_j^2S_{2,\,i-1}^{(2)}-$$
$$c_j'^3(6c_j^2+3c_j+1)[g_2(t_{i-1})-g_1'(t_{i-1})]+[g_2(t_{i-1+c_j})-g_1'(t_{i-1+c_j})]- \tag{2.18}$$
$$c_j^3(6\bar{c}_j^2+3\bar{c}_j+1)[g_2(t_i)-g_1'(t_i)] \quad , \quad j=1,2\,.$$

From the equations (2.9) and (2.18), we get the following recurrence formula:

$$\mathrm{A}_2\,\underline{S}_{2,\,i}=\mathrm{B}_2\,\underline{S}_{2,\,i-1}+\mathrm{D}_2\,\underline{G}_{2,\,i}\,,\; i=1(1)\mathrm{N}, \tag{2.19}$$

where

$$\mathrm{A}_2=\begin{bmatrix}-c_1^3(3\bar{c}_1^2+\bar{c}_1) & \tfrac{1}{2}c_1^3\,\bar{c}_1^2 & 0 & 0\\ -c_2^3(3\bar{c}_2^2+\bar{c}_2) & \tfrac{1}{2}c_2^3\bar{c}_2^2 & 0 & 0\\ 0 & 0 & -c_1^3(\bar{c}_1+3\bar{c}_1^2) & \tfrac{1}{2}c_1^3\,\bar{c}_1^2\\ 0 & 0 & -c_2^3(\bar{c}_2+3\bar{c}_2^2) & \tfrac{1}{2}c_2^3\,\bar{c}_2^2\end{bmatrix}=\left[\begin{array}{c:c}\mathrm{A}_1 & 0\\ \hdashline 0 & \mathrm{A}_1\end{array}\right],$$

$$\mathrm{B}_2=\begin{bmatrix}-\bar{c}_1^3(3c_1^2+c_1) & -\tfrac{1}{2}c_1^3\,\bar{c}_1^2 & 0 & 0\\ -\bar{c}_2^3(3c_2^2+c_2) & -\tfrac{1}{2}c_2^3\bar{c}_2^2 & 0 & 0\\ 0 & 0 & -\bar{c}_1^3(c_1+3c_1^2) & -\tfrac{1}{2}\,\bar{c}_1^3c_1^2\\ 0 & 0 & -\bar{c}_2^3(c_2+3c_2^2) & -\tfrac{1}{2}\bar{c}_2^3\,c_2^2\end{bmatrix}=\left[\begin{array}{c:c}\mathrm{B}_1 & 0\\ \hdashline 0 & \mathrm{B}_1\end{array}\right],$$

$$\mathrm{D}_2=\begin{bmatrix}-\bar{c}_1^3(6c_1^2+3c_1+1) & 1 & 0 & -c_1^3(6c_1^2+3c_1+1) & 0 & 0 & 0 & 0\\ -\bar{c}_2^3(6c_2^2+3c_2+1) & 0 & 1 & -c_2^3(6c_2^2+3c_2+1) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\bar{c}_1^3(6c_1^2+3c_1+1) & 1 & 0 & -c_1^3(6c_1^2+3c_1+1)\\ 0 & 0 & 0 & 0 & -\bar{c}_2^3(6c_2^2+3c_2+1) & 0 & 1 & -c_2^3(6c_2^2+3c_2+1)\end{bmatrix}$$
$$=\left[\begin{array}{c:c}\mathrm{D}_1 & \theta\\ \hdashline \theta & \mathrm{D}_1\end{array}\right],$$

$$\underline{S}_{2,\,i}=(S_{1,\,i}^{(1)},S_{1,\,i}^{(2)},S_{2,\,i}^{(1)},S_{2,\,i}^{(2)})^T,$$
$$\underline{S}_{2,\,i-1}=(S_{1,\,i-1}^{(1)},S_{1,\,i-1}^{(2)},,S_{2,\,i-1}^{(1)},S_{2,\,i-1}^{(2)})^T,$$
$$\underline{G}_{2,\,i}=(G_{1,\,i-1},G_{1,\,i-1+c_1},G_{1,\,i-1+c_2},G_{1,\,i},G_{2,\,i-1},G_{2,\,i-1+c_1},G_{2,\,i-1+c_2},G_{2,\,i})^T.$$
$$G_{1,i-1}=g_1(t_{i-1}),G_{1,i-1+c_1}=g_1(t_{i-1+c_1}),G_{1,i-1+c_2}=g_1(t_{i-1+c_2}),G_{1,i}=g_1(t_i),$$
$$G_{2,i-1}=g_2(t_{i-1})-g_1'(t_{i-1}),G_{2,i-1+c_1}=g_2(t_{i-1+c_1})-g_1'(t_{i-1+c_1}),$$
$$G_{2,i-1+c_2}=g_2(t_{i-1+c_2})-g_1'(t_{i-1+c_2}),G_{2,i}=g_2(t_i)-g_1'(t_i),$$

moreover, the 2×2 matrix 0 and the 4×8 matrix θ all elements are zeros, and the matrices A_1, B_1, D_1 are in the relation (2.10).

Multiplying (2.19) by A_2^{-1}, we get

$$\underline{S}_{2,\,i}=\tilde{\mathrm{A}}_2\,\underline{S}_{2,\,i-1}+\mathrm{A}_2^{-1}\mathrm{D}_2\,\underline{g}_{2,\,i}, \tag{2.20}$$

where $\tilde{\mathrm{A}}_2=\mathrm{A}_2^{-1}\mathrm{B}_2.$

We can find by using *Mahtematica Program* that $\tilde{\mathrm{A}}_2$ has the form

$$\tilde{A}_2 = \left[\begin{array}{c:c} \tilde{A}_1 & 0 \\ \hdashline 0 & \tilde{A}_1 \end{array}\right],$$ where, the 2×2 matrix 0 and $\tilde{A}_1$ as above.

If $0 < c_1 < c_2 < 1$, then $\tilde{A}_2$ exists because $|\tilde{A}_2| = \frac{(1-c_1)^4(1-c_2)^4}{c_1^4 c_2^4} \neq 0$.

3. The Methods QSCMs for Higher-Index Systems.

In general, suppose that QSCMs are applied to index-*v*, then (2.6) becomes:

$$S_1(t_{i-1+c_j}) = g_1(t_{i-1+c_j}), \tag{2.21a}$$

$$S_2(t_{i-1+c_j}) = g_2(t_{i-1+c_j}) - g_1'(t_{i-1+c_j}), \tag{2.21b}$$

$$\vdots$$

$$S_\nu(t_{i-1+c_j}) = g_\nu(t_{i-1+c_j}) + \sum_{r=1}^{\nu-1}(-1)^{\nu-r} g_r^{(\nu-r)}(t_{i-1+c_j}),\ j = 1(1)3,\ i = 1(1)N. \tag{2.21c}$$

Using the approximation (1.3)-(1.4) to (2.21a)-(2.21c) we get

$$\underline{S}_{\nu,i} = \tilde{A}_v \underline{S}_{\nu,i-1} + A_v^{-1} D_v \underline{g}_{\nu,i} \tag{2.22}$$

where to this end, $\tilde{A}_v$ can be found after tedious calculations, as

$$\tilde{A}_v = \left[\begin{array}{c:c:c:c} \tilde{A}_1 & 0 & \cdots & 0 \\ \hdashline 0 & \tilde{A}_1 & \ddots & \vdots \\ \hdashline \vdots & \cdots & \ddots & 0 \\ \hdashline 0 & \cdots & 0 & \tilde{A}_1 \end{array}\right],$$

where $\tilde{A}_v$ is a $2v \times 2v$ matrix, and which yields the following Corollary.

<u>**Corollary 3**</u>. The QSCMs applied to index-*v* systems DAEs are stable if $|\mu_1|,|\mu_2| \leq 1$, where μ_1, μ_2 are the eigenvalues of the matrix $\tilde{A}_1$ (2.11b).

Proof. Note first that If $0 < c_1 < c_2 < 1$, then $\tilde{A}_2$ is existed because $|\tilde{A}_v| = \frac{(1-c_1)^{2v}(1-c_2)^{2v}}{c_1^{2v} c_2^{2v}} \neq 0$. Since $\tilde{A}_v$ has the same eigenvalues of $\tilde{A}_1$ with multiplicity *v*, then according to Corollary 1 we find that two eigenvalues satisfy (2.13) for the same c_1, c_2 listed in Table1. □

Finally, for algebraic subsystem of index-*v*, the local error satisfies

$$\underset{\sim}{\tilde{d}}_{i,\nu}=\begin{bmatrix} h\,y_1'(t_i)\\ h^2 y_1''(t_i)\\ \vdots \\ h\,y_\nu'(t_i)\\ h^2 y_\nu''(t_i)\end{bmatrix}-\mathrm{A}_v^{-1}\mathrm{B}_v\begin{bmatrix} h\,y_1'(t_{i-1})\\ h^2 y_1''(t_{i-1})\\ \vdots \\ h\,y_\nu'(t_{i-1})\\ h^2 y_\nu''(t_{i-1})\end{bmatrix}-\mathrm{A}_v^{-1}\mathrm{D}_v\begin{bmatrix} g_1(t_{i-1})\\ g_1(t_{i-1+c_1})\\ g_1(t_{i-1+c_2})\\ g_1(t_i)\\ \vdots \\ g_\nu(t_{i-1})\\ g_\nu(t_{i-1+c_1})\\ g_\nu(t_{i-1+c_2})\\ g_\nu(t_i)\end{bmatrix}.$$

Using Taylor's expansion, we get

$$g_1(t)=y_1(t)=\sum_{r=0}^{5}\frac{h^r}{r!}y_1^{(r)}(t_{i-1})T^r+O(h^6), t\in[t_{i-1},t_i],\ T\in[0,1],\ y_1\in C^7[a,b]\ ,$$

$$g_2(t)=y_2(t)+y_1'(t)\equiv\sum_{r=0}^{5}\frac{h^r}{r!}y_2^{(r)}(t_{i-1})T^r+O(h^6)+\sum_{r=0}^{4}\frac{h^r}{r!}y_1^{(r+1)}(t_{i-1})T^r+O(h^5),$$

$$\vdots$$

$$g_v(t)=y_v(t)-(-1)^{v-1}y_{v-1}'(t)$$

$$=\sum_{r=0}^{5}\frac{h^r}{r!}y_v^{(r)}(t_{i-1})T^r+O(h^6)-(-1)^{v-1}\sum_{r=0}^{4}\frac{h^r}{r!}y_{v-1}^{(r+1)}(t_{i-1})T^r+O(h^5)\quad ,v=1,2,...$$

Thus, the local error is given by

$$g_1(t)-y_1(t)\equiv O(h^6),$$

$$g_2(t)-y_2(t)-y_1'(t)\equiv O(h^5),$$

$$\vdots$$

$$g_v(t)-y_v(t)+(-1)^{v-1}y_{v-1}'(t)\equiv O(h^5)$$

We observe that the methods QSCMs applied to systems with index greater than one are exact for polynomials of degree ≤ 4 , we deduce according to Definitions 2 that the methods are thus consistent and are of order **four** for all c_1, c_2 given in Table1.

Strict Stability:

Before we can get started, we need the following definition.

Definition 3 [15]. The QSCMs (1.6)-(1.7) applied to nonlinear systems of DAEs (1.2) are strictly stable if the difference between perturbed spline collocation methods step,

$$F(t_{i-1+c_j}, Z(t_{i-1+c_j})+\delta_{\nu,i}^{(k)},\ Z'(t_{i-1+c_j}))=0\ ,\ j,k=1(1)3\ ,\ i=1(1)\mathrm{N}\ , \qquad (3.1)$$

where $Z_0=S_0+\delta_0^{(0)}$, and $\left\|\delta_{v,i}^{(k)}\right\|\le\Delta_v$, $k=0(1)3$, and unperturbed spline collocation methods step (1.6)-(1.7), satisfy $\left\|Z(t_{i-1+c_j})-S(t_{i-1+c_j})\right\|\le K_0\Delta_v$, j=1(1)3, i=1(1)N, where $0<h\le h_0$ and K_0, h_0 are constants depending only on the method and the DAEs.

We now solve (2.5) by the perturbed spline collocation methods:

$$M\,Z'(t_{i-1+c_j}) + Z(t_{i-1+c_j}) - \delta^{(k)}_{\nu,i} = g(t_{i-1+c_j})\ ,\ j,k = 1(1)3,$$

where $Z' = (z'_1, z'_2, \ldots, z'_v)^T$, $Z = (z_1, z_2, \ldots, z_v)^T$.

Then, we have

$$\underline{Z}_{\nu,i} = \tilde{A}_v\,\underline{Z}_{\nu,i-1} + A_v^{-1} D_v\,\underline{g}_{\nu,i} + \underline{\delta}_{\nu,i}\,, \quad (3.2)$$

where the perturbations $\underline{\delta}_{\nu,\,i} = (\delta^{(1)}_{1,\,i}, \delta^{(2)}_{1,\,i}, \ldots, \delta^{(1)}_{\nu,\,i}, \delta^{(2)}_{\nu,\,i})^T$ satisfy $\left\|\underline{\delta}_{\nu,\,i}\right\| \le \Delta_\nu$,

$$\underline{Z}_{\nu,\,i} = (Z^{(1)}_{1,\,i}, Z^{(2)}_{1,\,i}, \ldots, Z^{(1)}_{\nu,\,i}, Z^{(2)}_{\nu,\,i})^T,$$

$$\underline{Z}_{\nu,\,i-1} = (Z^{(1)}_{1,\,i-1}, Z^{(2)}_{1,\,i-1}, \ldots, Z^{(1)}_{\nu,\,i-1}, Z^{(2)}_{\nu,\,i-1})^T.$$

Subtracting (3.2) from the corresponding expressions for the unperturbed solution (2.22), and letting $\underline{E}_{\nu,i} = \underline{Z}_{\nu,i} - \underline{S}_{\nu,i}$, we obtain,

$$\underline{E}_{\nu,i} = \tilde{A}_v\,\underline{E}_{\nu,i-1} + \underline{\delta}_{\nu,i}\ . \quad (3.3)$$

Using $\|.\|_\infty$, we have from (3.3)

$$\|\underline{E}_{\nu,i}\| \le R_\nu \|\underline{E}_{\nu,i-1}\| + \Delta_\nu\,, \quad (3.4)$$

where

$$R_\nu = \|\tilde{A}_\nu\| \text{ and } \|\underline{\delta}_{\nu,i}\| \le \Delta_\nu\,,$$

Inequality (3.4) is defined recursively by

$$\|\underline{E}_{\nu,i}\| \le R_\nu^i \|\underline{E}_{\nu,0}\| + \sum_{k=0}^{i-1} R_\nu^{\,k} \Delta_\nu\,,\ i=1(1)\mathrm{N}.$$

which can be rewritten in the form

$$\|\underline{E}_{\nu,i}\| \le R_\nu^{\,i} \|\underline{E}_{\nu,0}\| + \frac{1-R_\nu^{\,i}}{1-R_\nu}\Delta_v\,,\ i=1(1)\mathrm{N}.$$

Note that $\lim\limits_{N\to\infty} \dfrac{1-R_\nu^{\,N}}{1-R_\nu} = \dfrac{1}{1-R_\nu}$ if $R_\nu < 1$. Thus, we have the following theorem.

<u>Theorem</u> 2: The QSCMs are strictly stable for index-*v* systems of DAEs (1.2) iff:

$$R_\nu = \|\tilde{A}_v\| < 1. \quad (3.5)$$

Proof. To prove that inequality (3.5) holds, we easily find that $R_\nu = \|\tilde{A}_\nu\|_\infty \equiv \|\tilde{A}_1\|_\infty = \max\limits_{1\le i\le 2} \sum_{j=1}^{2} |\,\tilde{a}^1_{i,j}\,|\ \ , v \ge 1$, where $\tilde{A}_1 = (\tilde{a}^1_{i\,j})$. Using Mathematia, we get the values of c_1, c_2 which satisfy the relation $R_\nu < 1$ in Table3. Moreover, for $R_\nu < 1$, we have $\underset{i\to\infty}{Lim} \|\underline{E}_{\nu,i}\| \le \|\underline{E}_{\nu,0}\| \underset{i\to\infty}{Lim} R_\nu^{\,i} + \Delta_v \underset{i\to\infty}{Lim} \dfrac{1-R_\nu^{\,i}}{1-R_\nu} = K_0 \Delta_v\,,$

where $K_0 = \dfrac{1}{1-R_\nu}$. This implies according to Definition 3 that the QSCMs applied for index-*v* systems are strictly stable. □

Table 3: **The values of** $c_1,\ c_2$ **which satisfy the relation** $R_v < 1$

$c_1 = 0.91,\ c_2 = 0.999$	$R_v = 0.922525$
$c_1 = 0.92,\ c_2 = 0.99$	$R_v = 0.822904$
$c_1 = 0.93,\ c_2 = 0.98$	$R_v = 0.932804$
$c_1 = 0.94,\ c_2 = 0.97$	$R_v = 0.939393$
$c_1 = 0.95,\ c_2 = 0.98$	$R_v = 0.705175$
$c_1 = 0.95,\ c_2 = 0.999$	$R_v = 0.49026$
$0.949 \le c_1 < c_2 < 1$	$R_v \le 0.978285$

Numerical Results:

The experiments below are designed to test the efficiency of the methods QSCMs when applied to differential-algebraic systems for both linear and nonlinear problems. All computations where made with programs *Mathematica* Version 5.0.0.0 and Turbo Pascal in double precision.

Problem 4.1: Consider index-2 Hessenberg DAEs [**8**],

$$\begin{cases} y' = t\,z^2 + w + g_1(t), \\ z' = t\,Exp(y) + t\,w + g_2(t), \qquad 0 \le t \le 1, \\ 0 = y + t\,z + g_3(t), \end{cases}$$

with $y(0) = z(0) = w(0) = 0$, where $g_1(t),\ g_2(t)$ and $g_3(t)$ are compatible to exact solutions, $y(t) = Ln(1+t)$, $z(t) = w(t) = \dfrac{1}{1+t}$. The results are given in Table 4.

Problem 4.2: Consider the problem having four differential equations and one algebraic equation **[3]**

$$\begin{aligned} x_1' &= -e^{x}\, x_1 + x_2 + x_4 + y - e^{-t} \\ x_2' &= -x_1 + x_2 - \sin(t)\, x_3 + y - \cos(t) \\ x_3' &= \sin(t)\, x_1 + x_3 + \sin(x)\, x_4 - \sin^2(x) - e^{-x}\sin(x) \\ x_4' &= \cos(t)\, x_2 + x_3 + \sin(t)\, x_4 - e^{-t}(1+\sin(t)) - \cos^2(t) - e^{t} \\ 0 &= x_1 \sin^2(t) + x_2 \cos^2(t) + (x_3 - e^{t})(\sin(t) + 2\cos(t)) \\ &\quad + \sin(t)(x_4 - e^{-t})(\sin(t) + \cos(t) - 1) - \sin^3(t) - \cos^3(t) \end{aligned}$$

The exact solution to this system is $x_1 = \sin(t)$, $x_2 = \cos(t)$, $x_3 = e^{t}$, $x_4 = e^{-t}$, and $y(t) = e^{t}\sin(t)$. It is easy to verify that system is index-3 for all *t*. The absolute error of the approximate solution gives in Table 5. Fig.(1) shows both the approximate solution and the exact solution of *y* over the interval $0 \le t \le 10$, using the step size *h*=0.1.

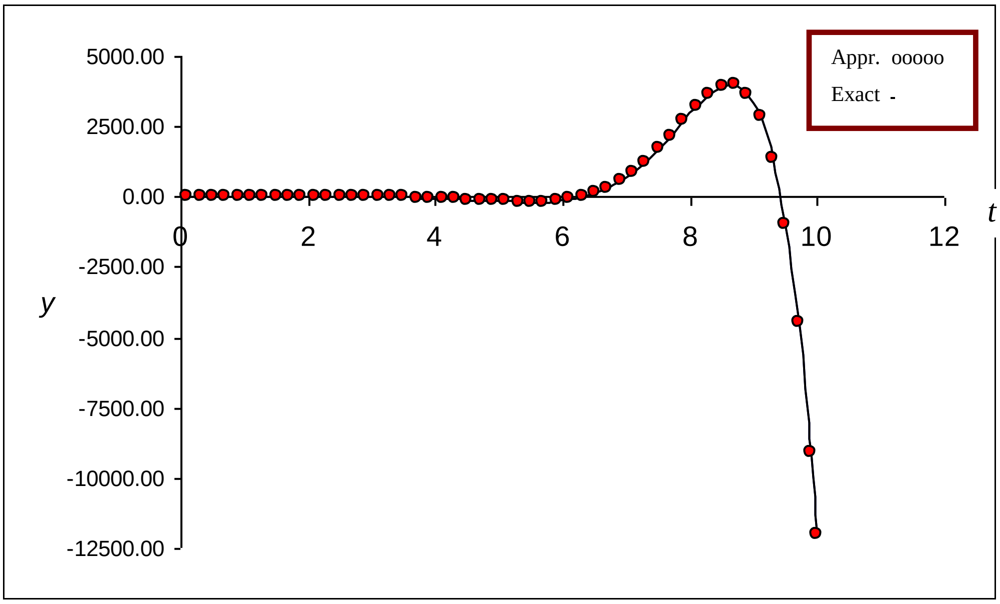

Fig.(1). Both the approximate solution and the exact solution of *y* in problem 4.2, for c_1=0.5, c_2=0.9998, and *h*=0.1.

Problem 4.3: Consider the nonlinear index-2 DAE [**1**]

$$y_1' = -y_1 + y_2 - \sin(t) - (1+2t),$$

$$y_2' = -y_1 y_3 \ ,$$

$$0 = y_1^2 + y_1 y_2 + y_1(-\sin(t) - 1 + 2t), \qquad t \in [0, 3],$$

subject to the initial condition $y_1(0)=1$, $y_2(0)=0$, $y_3(0)=-1$. The exact solution is $y_1(t)=1-2t$, $y_2(t)=\sin(t)$, $y_3 = -\cos(t)/(1-2t)$. A singularity is located at $t=\frac{1}{2}$. Using this problem, we test the spline methods formulations in Section 1. We list the computational results in Table6. Clearly, the spline methods work well for (c_1=0.57, c_2=0.9998), and (c_1=0.65, c_2=0.999) while Baumgarte's method [1] blows up upon hitting the singularity. In Fig.(2), we have plotted both the approximate solution and the exact solution of y_3 over the interval $0 \le t \le 3$, using the step size h=3/40, for c_1=0.6, c_2=0.99.

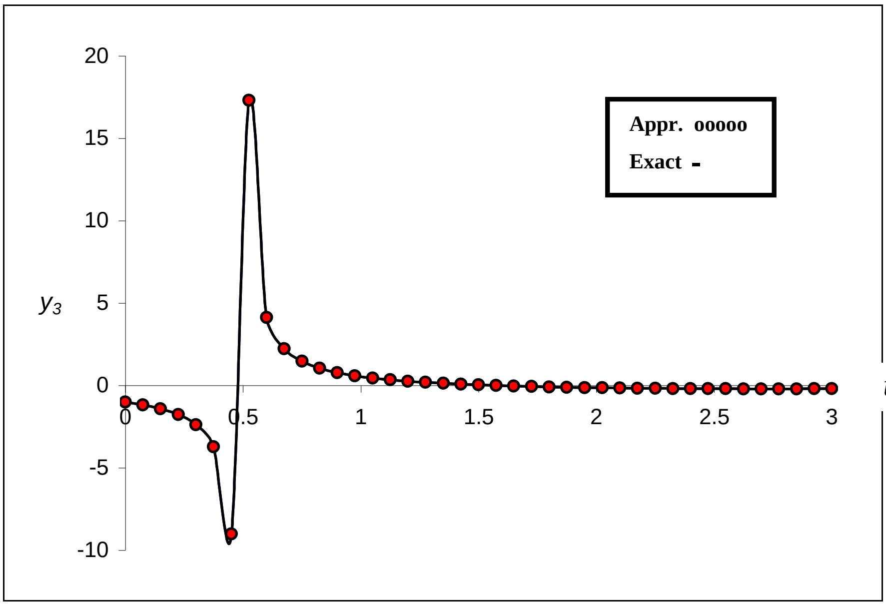

Fig.(2). Both the approximate solution and the exact solution of y_3 in problem 4.3, for c_1=0.6, c_2=0.99, and h=3/40.

Problem 4.4: Consider the nonlinear index-4 DAE

$$y_1' - y_2 = 0,$$
$$y_2' - y_3 = 0\,,$$
$$y_3' - y_4 = 0\,,$$
$$y_1 - Sin(t) = 0, \qquad t \in [0, 10],$$

subject to the initial conditions $y_1(0)=0$, $y_2(0)=1$, $y_3(0)=0$, $y_4(0)=-1$. The exact solution is $y_1(t)=sin(t)$, $y_2(t)=\cos(t)$, $y_3(t)=-\sin(t)$, $y_4(t)=-\cos(t)$. We show the computational results in Table7. Fig.(3) explains the approximate solutions and the exact solutions of y_1, y_2, y_3, y_4 by c_1=0.53, c_2=0.994, and h=0.05.

Table 4: **The global errors for the solution of problem 4.1** [**8**].

t	modified Adomian decomposition method [8]			Present method for c_1=0.5, c_2=0.9998		
	δy	δz	δw	δy	δz	δw
0.1	8.98056E-10	1.20912E-10	1.77933E-8	6.97075E-11	6.97075E-10	8.61539E-10
0.2	4.63777E-7	1.16056E-7	8.92432E-6	2.06748E-10	1.03374E-9	1.01819E-9
0.3	1.79601E-5	6.31386E-6	3.32789E-4	3.52903E-10	1.17634E-9	1.31223E-9
0.4	2.40711E-4	1.06358E-4	4.25255E-3	4.82894E-10	1.20723E-9	1.12924E-9
0.5	1.80331E-3	9.44037E-4	3.0025E-2	5.83259E-10	1.16651E-9	1.16546E-9
0.6	9.34963E-3	5.59329E-3	1.44694E-1	6.48861E-10	1.08143E-9	8.14168E-10
0.7	3.75984E-2	2.50912E-2	5.3171E-1	6.79883E-10	9.71262E-10	7.27947E-10
0.8	1.25531E-1	9.18675E-2	1.58743	6.76113E-10	8.45141E-10	3.43036E-10
0.9	3.63586E-1	2.88128E-1	3.99439	6.47592E-10	7.19547E-10	2.55590E-10
1.0	9.41377E-1	8.00027E-1	8.62456	5.93930E-10	5.93930E-10	6.03585E-11

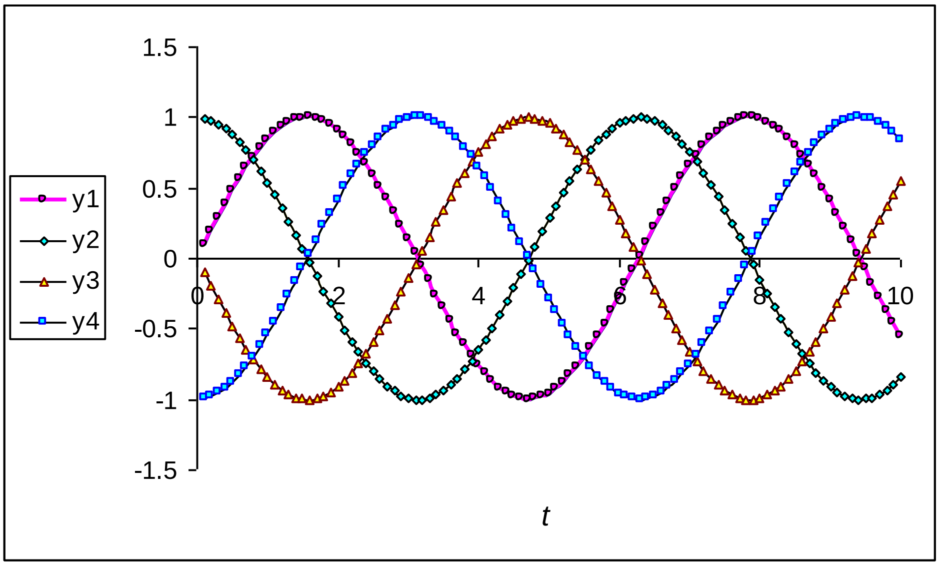

Fig.(3). The approximate solutions and the exact solutions of y_1 , y_2 , y_3, y_4 by c_1=0.53, c_2=0.994, and h=0.05, in problem 4.4.

Table 5: The absolute error for the solution of Problem 4.2 [3].

t	Present method (c_1=0.5, c_2=0.99), using the step size h=0.1.				
	δx_1	δx_2	δx_3	δx_4	δy
1.0	1.6254E-12	1.7312E-12	1.4800E-14	1.9253E-12	3.7635E-11
2.0	1.9271E-12	1.0742E-11	1.6891E-12	8.6127E-13	1.0713E-10
3.0	2.5514E-11	3.9746E-11	2.1647E-11	5.2040E-12	1.8240E-10
4.0	1.9124E-11	1.4622E-09	1.2677E-10	2.0503E-10	4.5873E-09
5.0	5.2136E-09	2.5915E-08	4.7350E-10	1.8017E-09	7.6844E-07
6.0	4.1878E-11	1.7231E-09	9.5689E-09	1.6561E-10	2.0305E-08
7.0	4.3640E-11	9.6056E-09	2.9558E-09	3.5497E-09	9.0305E-08
8.0	2.4179E-09	1.0691E-07	3.7785E-09	1.6413E-08	7.6251E-06
9.0	5.1095E-12	5.6267E-08	2.7056E-08	1.3849E-08	1.8266E-06
10.0	2.3257E-11	2.2294E-07	5.8476E-08	2.0865E-08	7.6798E-06

Table 6: The absolute error for the solution of problem 4.3 [1].

Time	SRM (α_1=0)[1]	Baumgarte's Method [1]	Present Methods	
			c_1=0.57, c_2=0.9998, h=1/15	c_1=0.65, c_2=0.999, h=0.03
0.1	0.40E-6	0.49E-7	0.285991E-14	----
0.2	--	--	0.415592E-13	0.14091E-13
0.3	0.25E-6	0.15E-6	0.807113E-13	----
0.4	--	--	0.178560E-12	0.18366E-13
0.5	0.14E-6	0.93E+1	0.634029E-12	----
0.6	--	--	0.279832E-12	0.88682E-13
0.7	0.46E-7	NAN	0.420595E-12	----
0.8	--	--	0.909076E-13	0.59696E-14
1.0	0.60E-7	NAN	0.770872E-13	0.19347E-14
2.0	---	---	0.400684E-13	0.19347E-14
3.0	---	---	0.149525E-13	0.17411E-14

Table 7: The absolute error for the solution of problem 4.4, with Index-4.

t	Present method (c_1=0.53, c_2=0.994), using the step size h=0.05.			
	δy_1	δy_2	δy_3	δy_4
1.0	0.0E+0000	5.3E-0013	3.5E-0009	6.89107801E-08
2.0	0.0E+0000	5.7E-0013	3.7E-0009	5.60850989E-08
3.0	0.0E+0000	8.9E-0014	5.8E-0010	1.56034565E-08
4.0	0.0E+0000	4.8E-0013	3.1E-0009	7.30371071E-08
5.0	0.0E+0000	6.0E-0013	4.0E-0009	6.04439542E-08
6.0	0.0E+0000	1.8E-0013	1.1E-0009	9.93085675E-09
7.0	0.0E+0000	4.1E-0013	2.7E-0009	7.27823161E-08
8.0	0.0E+0000	6.2E-0013	4.1E-0009	7.14150049E-08
9.0	0.0E+0000	2.6E-0013	1.7E-0009	8.14835006E-09
10.0	0.0E+0000	3.4E-0013	2.2E-0009	5.99824012E-08

Conclusions and Recommendations:

A collocation approach that produces a family of Quintic Spline Collocation Methods has been described for the approximate solution of problems in higher index differential-algebraic equations. The presented methods when applied to systems with index greater than one are consistent and are of order four for some c_1, c_2 given in Table1. The comparisons of our numerical results with other methods show that our results are better in accuracy than other methods. (see, Tables 4,6). The presented methods if applied to higher index differential-algebraic equations are accurate for solving problems, which have oscillatory solutions (see, Table7, Fig(3)).

Finally, we recommend the following:

Studying the QSCMs methods for solving boundary value problems of higher index algebraic-differential equations.

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