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On Lie Groups and P- two Norm Algebras

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□ ABSTRACT □

The objective of this paper is studying *Lie* group *G(X)* on algebra *X* and define two norms in this algebra and give some properties of two-norm algebras which are topological, and relate some concepts with the topological notions introduced by A. Alexiwicz [1].

Key Words: Two-Norm Algebras, Lie group.

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دراسة في زمر لي والجبور ثنائية النظيم

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□ ملخّص □

إن الهدف من هذا المقال هو دراسة زمرة لي $G(X)$ في الجبر X وتعريف نظيمين في هذا الجبر ودراسة بعض الخصائص للجبور ثنائية النظيم التي هي بحد ذاتها فضاء تبولوجي وربط بعض المفاهيم الجبرية بالمفاهيم التبولوجية وأن هذه الدراسة تعتبر تعميما للجبور ثنائية النظيم المعرفة من خلال [1] A. Alexiwicz حيث بنيت في هذا المقال على زمر لي.

الكلمات المفتاحية: الجبور ثنائية النظيم، زمرة لي.

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Introduction:

A vector space or algebra over a field of real or complex numbers will be denote by X . For the sets A and B in X we write

$$\alpha A = \{\alpha a; a \in A\},\ A.B = \{ab;\ a \in A,\ b \in B\},\ A^{-1} = \{a^{-1};\ a \in A\}$$

A topological space T is called Hausdorff if for any two distinct points $p1$; $p2 \in M$ there exists open sets $U1$;$U2 \in T$ with

$p1 \in U1$; $p2 \in U2$; $U1 \cap U2 = \phi$.

A map between topological spaces is called continous if the preimage of any open set is again open.

The important of research and its objectives:

The object of this paper is to generalize the notation of two-norm algebras introduced by A.Alexiewicz[1]. First of all we consider topologies in algebras which arise from two p-homogeneous norms.

Research methods and materials:

Let M be a manifold ,An open chart on M is a pair (U;□), where U is an open subset of M and □□is a homeomorphism of U onto an open subset of R^n.

Definition 1:[2]. A smooth manifold M is an n-dimensional Hausdorff space where in the neighborhood of any point $p \in M$ there exists a chart of n-dimension.

Definition 2:[2] A Lie group is a group that is also a smooth manifold such that the multiplication map:

$$\mu : \underset{(g,h)\ \to\ gh}{G \times G \to G}$$

$$\xi : \underset{g \to g^{-1}}{G \to G}$$

Are smooth. Space with a transitive G action for a Lie group G are called as is a homogeneous space. $S^2 = \{x \in R^3;\ |x| = 1\}$ homogeneous space. Write a

We shall denote by e the unit element of the algebra if existing. $G(X)$ and $G_0(X)$ will denote the multiplicative Lie group of the algebra X and the set of quasi invertible elements. $\gamma(\tau)$ will denote the neighborhood filter of zero for the topology τ.

Definition 3.[3].The function $\| \ \|: X \to R$ is said to be p- *homogeneous norm* or shortly p-norm where $0<p\leq 1$ if $\| . \|$ satisfy the following conditions:

(1) $\|x\|=0$ iff $x=0$

(2) $\|x+y\| \leq \|x\| + \|y\|$

(3) $\|\alpha x\| = |\alpha|^p \|x\|$

And $(X, \| \ \|)$ is said to be topological vector spaces or *t.v.s.* has a neighborhood basis of zero composed of bounded set.

Definition 4.[3]: The triplet $(X, \| \ \|, \| \ \|_0)$ is said to be *p-two- norm space* if X is a vector space and $\| \ \|, \| \ \|_0$ are two p- homogeneous norms, the first being finer then the second one. This is the case if and only if there exists a constant k such that $\|x\|_0 \leq k\|x\|$ for every $x \in X$.

Let τ_0 denote the topology generated in X by the metric $\rho(x,y) = \|x-y\|$,let $S_n=\{x \in X;\ \|x\| \leq n\}$. By τ_∞ we shall denote the finest vector topology on X which induced on S a topology coarser than τ_0. such topology exist.

Dsefinition 5. [3]: Let $(X, \|\ \|,\|\ \|_0)$ be p-two- norm space, suppose that in X defined multiplication of elements making X together with the vector operation an algebra . If multiplication is continuous for the topology τ_∞ then $(X, \|\ \|,\|\ \|_0)$ is called p-two norm algebra, the topology τ_∞ is then called multiplicative

Theorem 1. The topology τ_∞ is multiplicative if and only if

(i) there exists a constant β such $\|xy\| \le \beta\|x\|\|y\|$

(ii) for every $U\in\gamma(\tau_0)$ there exists a $V\in\gamma(\tau_0)$ such that $S(V\cap S)\subset U$, $(V\cap S)S\subset U$.

Proof: Necessity. The set S is τ_0 –bounded, for if $x_n\in S$,$\alpha_n\to 0$, then

$$\|\alpha_n x_n\| = |\alpha_n|^p \|x_n\|\to 0 .$$

The set SS is also τ_0 –bounded, Indeed let $\|x_n\|\le 1$, $\|y_n\|\le 1, \alpha_n\to 0$, then

$$\|\alpha_n x_n y_n\| = \|\sqrt{\alpha_n}\, x_n \sqrt{\alpha_n}\, y_n\| \to 0$$

Let us prove the necessity of the condition (i).

1-Suppose, if possible, that there exist x_n, y_n such that

$\|x_n\|\le 1$, $\|y_n\|\le 1$, $\|x_n y_n\|>n\|x_n\|\|y_n\|$,Then $\|x_n\|\ne 0\ne \|y_n\|$ and

$$\left\| \frac{x_n}{n^{1/2p}\|x_n\|^{1/p}} \frac{y_n}{n^{1/2p}\|y_n\|^{1/p}} \right\| > 1$$

This is impossible, since $\|x_n / \|x_n\|^{1/p}\| = \|y_n / \|y_n\|^{1/p}\| = 1$

We now prove that the condition (ii)is necessary. Observe first that the set S is τ_0 –bounded. Indeed, let $x_n\in S$,$\alpha_n\to 0$, then $\|\alpha_n x_n\| = |\alpha_n|^p \|x_n\|\to 0$,therefore $\|\alpha_n x_n\|_0 \to 0$

Next observe that the topology τ_0 is coarser than τ_∞. To see this choose a sequence (U_n) of τ_0-neighberhoods of zero, such that $U_1 + U_1\subset U$, $U_{n+1} + U_{n+1}\subset U_n$ for $n=1,2,3,...$

Then $V := \sum_{i=1}^{\infty} U_n \cap S$ *is in* $\gamma(\tau_\infty)$ *and* $V \subset \sum_{i=1}^{\infty} U_n \subset U.$

Now let U be in $\gamma(\tau_0)$. Since the topology τ_∞ is multiplicative and $\tau_0\le \tau_\infty$, so there exists $W\in\gamma(\tau_\infty)$ such that $W\subset U$. Thus $WW=\sum V_n\cap S$ where $V_n\in\gamma(\tau_0)$ and therefore $(V_1\cap S)W\subset U$. Since the set W is τ_0-bounded, $\alpha W \subset V_1$ for some $\alpha > 0$ and it is enough to choose $V = min(\alpha,\alpha^{-1})V_1$. The second part of (ii) is proved similarly.

2-The sufficiency of both conditions. Since $\|xy\|\le\beta\|x\|\|y\|$ introducing a new p-norm $\|x\|'=\sqrt{\beta}\|x\|$ we obtain a sub- multiplicative p-norm equivalent to $\|x\|$ for which the unit ball equals $\beta^{-1/2p}S$, and we can set $\sum U_n\cap \beta^{-1/2p}S$ give also the neighborhood basis of zero for τ_∞. So we can suppose freely that $\|xy\| \le \|x\|\,\|y\|$ and therefore that $SS \subset S$

II- On continuity of the inverse

Let the algebra X has the unit element e and denote by G(X) the multiplicative Lie group of X. We shall say that the inverse is ϑ- continuous if:

$$(a)\ \ x_n \xrightarrow{\rho} e \textit{ implies that almost all } x_n \textit{ are in } G(X)$$

$$(b)\ \textit{if}\ \ x_n \in G(X),\ x_n \xrightarrow{\rho} e, \textit{ then } x_n^{-1} \xrightarrow{\rho} e$$

From this condition it follows that if $x_n \xrightarrow{\rho} x_0 \in G(X)$, then almost all x_n are in G(X) and $x_{n+k}^{-1} \xrightarrow{\rho} x_0^{-1}$ for some k in N

The condition (b) is equivalent to the following one:

$$(b')\ \text{Let } x_n \in G(X), x_n \xrightarrow{\rho} e\ \ \text{, then} \sup_n \|x_n^{-1}\| < \infty$$

The necessity of this condition being obvious, we only need to prove its sufficiency.

So let $x_n \in G(X), x_n \xrightarrow{\rho} 1$. We distinguish two cases.

1. There exists a $\delta > 0$ such that $\| x_n - 1 \| \geq \delta$ for all n. Then

$$\| x_n^{-1} - 1 \|_0 = \| x_n^{-1}(1 - x_n) \| = \| x_n^{-1} \| \| 1 - x_n \| \left\| \frac{x_n^{-1}}{\| x_n^{-1} \|^{1/p}} \frac{1 - x_n}{\| 1 - x_n \|^{1/p}} \right\|_0 \leq$$

$$\leq \sup_n \| x_n^{-1} \| \sup_n \| 1 - x_n \| \varepsilon$$

Provided that $\| (1 - x_n) / \| 1 - x_n \|^{1/p} \|_0 \leq \eta(\varepsilon)$ *i.e when* $\| 1 - x_n \|_0 \leq \eta(\varepsilon)/\delta$

2. $\lim_{n \to \infty} \| x_n - 1 \| = 0$, then $\vartheta > 0$ being small enough, $\| x_n - 1 \| \leq \vartheta$ implies $\| x_n^{-1} - 1 \| < \varepsilon$

When the inverse is ρ - continuous, there exist two functions $f, g : R^+ \to R^+$ *such that* :

$(c) \; \| x \| \leq q, \| x \|_0 \leq f(q)$ *implies* $1 + x \in G(X)$

$(d) \; \| x \| \leq q, \| x \|_0 \leq f(q)$ *implies* $\| (1 + x)^{-1} \| \leq g(q)$

Definition 6.[4] A topological algebra (X, τ) is called *inverse continuous* if the multiplicative lie group $G(X)$ is τ-open and the map $x \to x^{-1}$ is τ-continuous on $G(X)$

Theorem 2. A p-two norm algebra (with unite) equipped with the Wiweger topology is inverse continuous if and only if the inverse is ρ-continuous.

Proof: The necessity being obvious, we prove the sufficiency.

1-So let us suppose that the inverse is ρ-continuous. First we prove that G(X) is τ_∞-open. It is enough to

show that $1 \in G(X)$. Let $\varpi_1 = f(1), \sigma_1 = g(1), \quad \varpi_{n+1} = \rho\left(\frac{f(\sigma_n)}{\sigma_n} \right), \sigma_{n+1} = \sigma_n (1 + \sigma_{n+1})$

We shall prove that:

(*) *if* $\| x_i \| \leq 1, \| x_i \|_0 \leq \varpi_1$, *then* $1 + x_1 + ... + x_n \in G(X)$ *and* $\| (1 + x_1 + ... + x_n)^{-1} \| \leq \sigma_n$

For $n = 1, 1 + x_1 \in G(X)$ *and* $\| (1 + x_1)^{-1} \| \leq g(1) = \sigma_1$. Suppose now that(*) holds for The element $x_1, ..., x_n$ and let us suppose that $x_1, ..., x_n, x_{n+1}$ fulfill the assumption (*).

Let, $v = 1 + x_1 + ... + x_n$, *then* $v \in G(X)$ *and* $\| (1 + v)^{-1} \| \leq \sigma_n$

Thus $w := 1 + x_1 + ... + x_n = v(1 + v^{-1} x_{n+1})$ *and* $\| v^{-1} x_{n+1} \| \leq \| v^{-1} \| \| x_{n+1} \| \leq \| v^{-1} \| \leq \sigma_n$.

Moreover

$$\| v^{-1} x_{n+1} \|_0 = \left\| \| v^{-1} \|^{1/p} \frac{v^{-1}}{\| v^{-1} \|^{1/p}} x_{n+1} \right\|_0 \leq \| v^{-1} \| \left\| \frac{v^{-1}}{\| v^{-1} \|^{1/p}} x_{n+1} \right\|_0$$

And since $\| v^{-1} / \| v^{-1} \|^{1/p} \| \leq 1, \| x_{n+1} \| \leq 1, \| x_{n+1} \|_0 \leq w_{n+1} = \rho(f(\sigma_n)/\sigma_n)$

We obtain by (3) $\| v^{-1} x_{n+1} \|_0 \leq \| v^{-1} \| \dfrac{f(\sigma_n)}{\sigma_n} \leq f(\sigma_n)$, *for* $\| v^{-1} \| \leq \sigma_n$.

Hence by (c) $1 + v^{-1} x_{n+1} \in G(X)$ and therefore $w \in G(X)$ and

$$\| w \| \leq \| v \| (\| 1 \| + \| v^{-1} x_{n+1} \|) \leq \sigma_n (1 + \sigma_n) = \sigma_{n+1}$$

From (*) it follows that $U_n = \{ x \in X; \| x \|_0 \leq w \}$, *then the set* $U := 1 + \sum_{n=1}^{\infty} U_n \cap S$

Which is a τ-neighborhood of 1. is contained in $G(X)$.

2-We now prove that the inverse is τ_∞ - continuous at 1. we shall consider two neighborhood bases for τ_∞ the first B_1 composed of sets of form $\sum_{n=1}^{\infty} P_n \cap S$

Where $P_n \in \gamma(\tau_0)$, the second B_2 composed of sets of form $\sum_{n=1}^{\infty} Q_n \cap \sigma_n(1+\sigma_n)S$

$Q_n \in \gamma(\tau_0)$. It is enough to show that for every U belongs to B_2 there exists V belongs to B_1 such that $1+V \in G(X),\ and\ (1+V)^{-1} \in 1+U.$

We can choose $Q_n = \{x \in X; \| x \|_0 \le \varepsilon_n\},\ \varepsilon_n > 0.$

Let $\eta_1 = \min(w_1, \rho(\varepsilon_1/\sigma_1)),\ \eta_n = \min(1, w_n, \rho(\varepsilon_n/\sigma_n\sigma_{n+1}))$

We shall show that for $\| x_i \| \le 1, \| x_i \|_0 \le \eta_i,\ i = 1,2,...,$ *we have*

$(I)\ 1 + x_1 + ... + x_n \in G(X)$

$(II)\ (1 + x_1 + ... + x_n)^{-1} = 1 + u_1 + ... + u_n; \| u_i \| \le \sigma_i(1+\sigma_i),\ \| u_i \|_0 \le \eta_i.$

From $\eta_i \le w_i$, (I) follows immediately.

For $n = 1, (1+x_1)^{-1} = 1 + u_1$, where $u_1 = (1+x_1)^{-1} - 1 = -(1+x_1)^{-1} x_1$, whence

$\| u_1 \| \le \| (1+x_1)^{-1} \| \| x_1 \| \le \sigma_1 \le \sigma_1(1+\sigma_1)$ *and*

$$\| u_1 \|_0 = \| (1+x_1)^{-1} x_1 \|_0 = \left\| \| (1+x_1)^{-1} \|^{1/p} \frac{(1+x_1)^{-1}}{\| (1+x_1)^{-1} \|^{1/p}} x_1 \right\| \le$$

$$\le \| (1+x_1)^{-1} \| \left\| \frac{(1+x_1)^{-1}}{\| (1+x_1)^{-1} \|^{1/p}} x_1 \right\|_0$$

And since $\left\| \frac{\| (1+x_1)^{-1} \|}{\| (1+x_1)^{-1} \|^{1/p}} \right\| \le 1, \| x_1 \| \le 1, \| x_1 \|_0 \le \rho(\varepsilon_1/\sigma_1)$ *we obtain by* (3)

$$\| (1+x_1)^{-1} \| \left\| \frac{(1+x_1)^{-1}}{\| (1+x_1)^{-1} \|^{1/p}} \right\|_0 \le \| (1+x_1)^{-1} \| \frac{\varepsilon_1}{\sigma_1} \le \sigma_1 \frac{\varepsilon_1}{\sigma_1} = \varepsilon_1$$

So (II) is true for n = 1. Suppose now it is valid for any set of n elements, and let

$\| x_i \| \le 1, \| x_i \|_0 \le \eta_i;\ i = 1,2,..., n+1,$ *Then* $x := 1 + x_1 + ... + x_n \in G(X),$

$x^{-1} = 1 + u_1 + ... + u_n; \| u_i \| \le \sigma_i(1+\sigma_i), \| u_i \|_0 \le \varepsilon_i$ *and*

$(1 + x_1 + ... + x_n)^{-1} = 1 + u_1 + ... + u_n + u_{n+1}$ *where*

$u_{n+1} = (1 + x_1 + ... + x_n + x_{n+1})^{-1} - (1 + x_1 + ... + x_n)^{-1} =$

$= -(1 + x_1 + ... + x_{n+1})^{-1} x_{n+1} (1 + x_1 + ... + x_n)^{-1}.$

Now

$\| u_{n+1} \| \le \| (1 + x_1 + ... + x_{n+1})^{-1} \| \| x_{n+1} \| \| (1 + x_1 + ... + x_n)^{-1} \| \le \sigma_n\sigma_{n+1},$

$u_{n+1} = v_1 x_{n+1} + v_2;\ v_1 = (1 + x_1 + ... + x_{n+1})^{-1},\ v_2 = (1 + x_1 + ... + x_n)^{-1}$

Whence

$$\| u_{n+1} \|_0 \le \| v_1 x_{n+1} v_2 \|_0 = \left\| \| v_1 \|^{1/p} \frac{v_1}{\| v_1 \|^{1/p}} x_{n+1} \| v_2 \|^{1/p} \frac{v_2}{\| v_2 \|^{1/p}} \right\|_0 \le$$

$$\le \| v_1 \| \| v_2 \| \left\| \frac{v_1}{\| v_1 \|^{1/p}} x_{n+1} \frac{v_2}{\| v_2 \|^{1/p}} \right\|_0$$

Since $\| v_1 / \| v_1 \|^{1/p} \| \le 1, \| v_2 / \| v_2 \|^{1/p} \| \le 1,$ *we get*

$$\| u_{n+1} \|_0 \le \| v_1 \| \| v_2 \| \frac{\varepsilon_n}{\sigma_n \sigma_{n+1}} \le \sigma_n \sigma_{n+1} \frac{\varepsilon_n}{\sigma_n \sigma_{n+1}} = \varepsilon_n$$

$P_n = \{ x \in X ; \| x_n \| \le \eta_n \}$ To obtain the desired result it is enough to choose

This concludes the proof.

Example: An example a p-two norm algebra may serve the algebra V^p for 0<p<1. The element of V^p are function x from <0,1> to R whose p-variation is finite.

The p-variation, $\mathrm{var}_p x$, of the function x is defined as the supremum of all the sum

$$\sum_{i=1}^{n} | x(t_i) - x(t_{i-1}) |^p ; \; 0 = t_0 \le t_1 \le ... \le t_n = 1$$

All three algebra operation are defined point wise. Let us introduce the p-norms

$$\| x \| = \| x(0) \|^p + \mathrm{var}_p (x)$$

$$\| x \|_0 = \sup \{ | x(t) |^p ; \; 0 \le t \le 1 \}$$

We obtain *p*-two norm spaces. From the obvious inequality

$$\| xy \| \le \| x \|_0 \| y \| + \| x \| \| y \|_0 \le 2 \| x \| \| y \|$$

$$\| xy \|_0 \le \| x \|_0 \| y \|_0$$

It follows that $(V^p, \| \|, \| \|_0)$ is a p-two-norm algebra.

The function equals to 1 its unit element. The sequence (x_n) of element V^p is $\mathrm{var}_p (x_n) < \infty$ *and* $x_n(t)$ *tend to* $x_0(t)$ *uniformly* γ-convergent to x_0 if and only if

. *on* $< 0,1 >$

A simple calculation show that if $\inf \{ | x(t) | ; 0 \le t \le 1 \} = \varepsilon > 0$, then $\mathrm{var}_p \frac{1}{x} \le \frac{\mathrm{var}_p x}{\varepsilon^{2p}}$,

If follows that the inverse is γ-continuous, and $G(V^p)$ consist of functions different from zero.

The neighborhood basis of zero for the topology τ_∞ consists of sets of form $\sum_{i=1}^{n} W_n$;

$W_n = \{ x \in V^p ; \; \mathrm{var}_p \, x \le 1, \; \| x \|_0 \le \varepsilon_n \}$ with some $\varepsilon_n > 0$.

Conclusion*:*

The problem arises as to the condition $\| \alpha x \| = | \alpha |^p \| x \|$ Characterizing p-normed spaces could be replaced by $\| \alpha x \| \le f(\alpha) \| x \|$ with some real function f, $\| \; \|$ being a Fréchet norm?

In this case we must have $f(\alpha\beta) \| x \| = \| \alpha\beta x \| = f(\alpha) \| \beta x \| = f(\alpha) f(\beta) \| x \|$, whence $f(\alpha\beta) = f(\alpha) f(\beta)$ for arbitrary α, β and $f(\alpha) > 0$ for $\alpha \ne o$. since the function $f \to \| f(x) \|$ is continuous, the function f also must be so.

it is well known that this is the case if and only if $f(x) = |x|^p$.

Also we can ask if the *p*-homogeneity of the norm might be replace $\| \alpha x \| \leq f(\alpha) \| x \|$ together with $\lim_{\alpha \to 0} f(\alpha) = 0$. This also gives nothing new, since in this case the ball $\{x \in X; \| x \| \leq 1\}$. must be bounded and the theorem of Aoki-Rolewicz the Fréchet norm $\| \ \|$ must be equivalent to *p*-norm.

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